Methods in Calculus 7B

1 a $f(x) = x^2 - 1,\ f(1) = 0,\ f'(x) = 2x$

$g(x) = x^2 + 3x - 4,\ g(1) = 0,\ g'(x) = 2x + 3$

$$\lim_{x\to 1}\frac{x^2-1}{x^2+3x-4} = \lim_{x\to 1}\frac{2x}{2x+3} = \frac{2}{5}$$

b $f(x) = x - 4,\ f(4) = 0,\ f'(x) = 1$

$g(x) = \sqrt{x} - 2,\ g(4) = 0,\ g'(x) = \frac{1}{2\sqrt{x}}$

$$\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4}\frac{1}{\frac{1}{2\sqrt{x}}} = 4$$

c $f(x) = \ln(x),\ \lim_{x\to\infty} f(x) = \infty,\ f'(x) = \frac{1}{x}$

$g(x) = x^2,\ \lim_{x\to\infty} g(x) = \infty,\ g'(x) = 2x$

$$\lim_{x\to\infty}\frac{\ln(x)}{x^2} = \lim_{x\to\infty}\frac{\frac{1}{x}}{2x} = 0$$

d $f(x) = x^2 - x,\ f(0) = 0,\ f'(x) = 2x - 1$

$g(x) = x^2 - \sin(\pi x),\ g(0) = 0,$

$g'(x) = 2x - \pi\cos(\pi x)$

$$\lim_{x\to 0}\frac{x^2-x}{x^2-\sin(\pi x)} = \lim_{x\to 0}\frac{2x-1}{2x-\pi\cos(\pi x)} = \frac{1}{\pi}$$

e $f(x) = e^{4x} - 4x - 1,\ f(0)$

$= 0,\ f'(x) = 4e^{4x} - 4$

$g(x) = e^x - \cos(x),\ g(0) = 0,$

$g'(x) = e^x + \sin(x)$

$$\lim_{x\to 0}\frac{e^{4x}-4x-1}{e^x-\cos(x)} = \lim_{x\to 0}\frac{4e^{4x}-4}{e^x+\sin(x)} = \frac{0}{1} = 0$$

f $f(x) = \arctan(4x),\ f(0) = 0$

$g(x) = \arctan(5x),\ g(0) = 0$

$f'(x) = \frac{4}{16x^2+1},\ g'(x) = \frac{5}{25x^2+1}$

$$\lim_{x\to 0}\frac{\arctan(4x)}{\arctan(5x)} = \lim_{x\to 0}\frac{\frac{4}{16x^2+1}}{\frac{5}{25x^2+1}} = \frac{4}{5}$$

2 a $f(x) = x\sin(x),\ f(0) = 0,$

$f'(x) = x\cos(x) + \sin(x)$

$g(x) = e^x - 1,\ g(0) = 0,\ g'(x) = e^x$

$$\lim_{x\to 0}\frac{x\sin(x)}{e^x-1} = \lim_{x\to 0}\frac{x\cos(x)+\sin(x)}{e^x} = \frac{0}{1} = 0$$

b $f(x) = \sqrt{x},\ f(0) = 0,\ f'(x) = \frac{1}{2\sqrt{x}}$

$g(x) = \tan(x),\ g(0) = 0,\ g'(x) = \sec^2(x)$

$$\lim_{x\to 0^+}\frac{\sqrt{x}}{\tan(x)} = \lim_{x\to 0^+}\frac{\frac{1}{2\sqrt{x}}}{\sec^2(x)} = \frac{\infty}{1} = \infty$$

$$\lim_{x\to 0^-}\frac{\sqrt{x}}{\tan(x)} = \lim_{x\to 0^-}\frac{\frac{1}{2\sqrt{x}}}{\sec^2(x)} = \frac{-i\infty}{1} = -i\infty$$

The limit does not exist.

c $f(x) = x^2 + x + 1,\ f'(x) = 2x + 1$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0,\ f''(x) = 2$

$g(x) = e^x = g'(x) = g''(x)$

$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} g'(x) = 0$

$$\lim_{x\to\infty}\frac{x^2+x+1}{e^x} = \lim_{x\to\infty}\frac{2x+1}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0$$

3 a $\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\log(x)} = e^{\lim_{x\to 0} x\log(x)}$

$$\lim_{x\to 0} x\ln(x) = \lim_{x\to 0}\frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$= \lim_{x\to 0}(-x) = 0$

as $-\lim_{x\to 0}\ln(x) = \lim_{x\to 0}\frac{1}{x} = \infty$

$\lim_{x\to 0} x^x = e^0 = 1$

b $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{1}{x}\ln(x)} = e^{\lim_{x\to\infty}\frac{1}{x}\ln(x)}$

$$\lim_{x\to\infty}\frac{\ln(x)}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0$$

as $\lim_{x\to\infty}\ln(x) = \lim_{x\to\infty} x = \infty$

$\lim_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1$

3 c $\lim_{x\to 0}(1-x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1-x)} = e^{\lim_{x\to 0}\frac{1}{x}\ln(1-x)}$

$\lim_{x\to 0}\frac{\ln(1-x)}{x} = \lim_{x\to 0}\frac{\frac{-1}{1-x}}{1} = -1$

as $\lim_{x\to 0}\ln(1-x) = \lim_{x\to 0} x = 0$

$\lim_{x\to 0}(1-x)^{\frac{1}{x}} = e^{-1}$

4 a $\frac{2x^2+x-1}{3x^2-2x-1} = \frac{2}{3} + \frac{x-1+\frac{4}{3}x+\frac{2}{3}}{3x^2-2x-1}$

$= \frac{2}{3} + \frac{1}{3}\frac{7x-1}{3x^2-2x-1} = \frac{2}{3} + \frac{B}{3x+1} + \frac{C}{x-1}$

$\frac{1}{3}(7x-1) = B(x-1) + C(3x+1)$

$B = \frac{5}{6}, C = \frac{1}{2}$

$\frac{2x^2+x-1}{3x^2-2x-1}$

$= \frac{2}{3} + \frac{1}{3}\frac{7x-1}{3x^2-2x-1} = \frac{2}{3} + \frac{5}{6(3x+1)} + \frac{1}{2(x-1)}$

b $\lim_{x\to\infty}\frac{2x^2+x-1}{3x^2-2x-1}$

$= \lim_{x\to\infty}\left(\frac{2}{3} + \frac{5}{6(3x+1)} + \frac{1}{2(x-1)}\right)$

$= \frac{2}{3}$

c $\lim_{x\to\infty}\frac{2x^2+x-1}{3x^2-2x-1} = \frac{2}{3}$

and

$\lim_{x\to\infty}\frac{4x+1}{6x-2} = \lim_{x\to\infty}\frac{4}{6} = \frac{2}{3}$

5 a The limit is not in indeterminate form as the limits of the numerator and denominator both exist and are not both 0

b $\lim_{x\to 3}\frac{x^2-5x+6}{4x} = \frac{0}{12} = 0$

6 a $f(x) = \cosh(x), f(0) = 1$

$g(x) = 2x^2, g(0) = 0$

Limit is not indeterminate, $\lim_{x\to\infty}\frac{\cosh(x)}{2x^2} = \infty$

6 b $f(x) = \cosh(x) - 1, f'(x) = \sinh(x)$

$f''(x) = \cosh(x)$

$g(x) = 2x^2, g'(x) = 4x, g''(x) = 4$

$f(0) = f'(0) = g(0) = g'(0) = 0$

$\lim_{x\to\infty}\frac{\cosh(x)-1}{2x^2} = \frac{f''(0)}{g''(0)} = \frac{1}{4}$

7 $f(x) = \sin^2(x), f'(x) = 2\cos(x)\sin(x)$

$f''(x) = 2\cos^2(x) - 2\sin^2(x)$

$f(0) = f'(0) = 0, f''(0) = 2$

$g(x) = x\tan(x), g'(x) = \tan(x) + x\sec^2(x)$

$g''(x) = 2\sec^2(x) - x\tan(x)\sec^2(x)$

$g(0) = g'(0) = 0, g''(0) = 2$

$\lim_{x\to 0}\frac{\sin^2(x)}{x\tan(x)} = \frac{f''(0)}{g''(0)} = 1$

8 $f(x) = x^2e^x, f'(x) = x^2e^x + 2xe^x$

$f''(x) = x^2e^x + 4xe^x + 2e^x$

$f(0) = f'(0) = 0, f''(0) = 2$

$g(x) = \tan^2(x), g'(x) = 2\tan(x)\sec^2(x)$

$g''(x) = 2\sec^4(x) + 4\tan^2(x)\sec^2(x)$

$g(0) = g'(0) = 0, g''(0) = 2$

$\lim_{x\to 0}\frac{x^2e^x}{\tan^2(x)} = \frac{f''(0)}{g''(0)} = 1$

9 $\lim_{x\to 0}\ln(x)\sin(x) = \lim_{x\to 0}\frac{\sin(x)}{(\ln(x))^{-1}}$

$f(x) = \sin(x), f'(x) = \cos(x)$

$g(x) = \frac{1}{\ln(x)}, g'(x) = \frac{1}{(\ln(x))^2}\frac{(-1)}{x}$

Applying L'Hospital's rule to $g'(x)$, simplifying and applying L'Hospital's rule again:

$\lim_{x\to 0}\frac{-\frac{1}{x}}{(\ln(x))^2} = \lim_{x\to 0}\frac{\frac{1}{x^2}}{2\ln(x)\frac{1}{x}} = \lim_{x\to 0}\frac{\frac{1}{x}}{2\ln(x)}$

$= \lim_{x\to 0}\frac{-\frac{1}{x^2}}{2\frac{1}{x}} = \lim_{x\to 0}\frac{-1}{2x} = -\infty$

$f'(0) = 1, g'(0) = -\infty, \lim_{x\to 0}\log(x)\sin(x) = 0$

10 $f(x)=\sqrt{x}-\sqrt{k}, f'(x)=\dfrac{1}{2\sqrt{x}}$

$g(x)=\sqrt[3]{x}-\sqrt[3]{k}, g'(x)=\dfrac{1}{3}x^{-\frac{2}{3}}$

$f(k)=g(k)=0$

$$\lim_{x\to k}\frac{\sqrt{x}-\sqrt{k}}{\sqrt[3]{x}-\sqrt[3]{k}}=\lim_{x\to k}\frac{\frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{3}x^{-\frac{2}{3}}}=\frac{3}{2}k^{\frac{1}{6}}$$

11 $\lim_{x\to 0}(\cos x)^{\frac{1}{x}}=e^{\lim_{x\to 0}\frac{1}{x}\ln(\cos x)}$

$\lim_{x\to 0}\frac{1}{x}\ln(\cos x)=\lim_{x\to 0}\dfrac{-\tan x}{1}=0$

as $\ln(\cos(0))=0$

$\lim_{x\to 0}(\cos x)^{\frac{1}{x}}=e^0=1$

12 $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$f(x)=\sin x$

$F(h)=\sin(x+h)-\sin x, F(0)=0$

$F'(h)=\cos(x+h), F'(0)=\cos x$

$G(h)=h, G(0)=0, G'(h)=1$

$f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\dfrac{\cos x}{1}=\cos x$

13 a $£1000\times 1.05^5=£1276.28$ (2 d.p.)

b $1+x=\left(1+\dfrac{0.1}{12}\right)^{12}$

$x=\left(1+\dfrac{0.1}{12}\right)^{12}-1\approx 0.1047$

Effective interest rate is approximately 10.47%

c $A_n(r)=A(1+\frac{r}{n})^n$

d $A_\infty(r)=\lim_{n\to\infty}A(1+\frac{r}{n})^n$

$=Ae^{\lim_{n\to\infty}n\ln(1+\frac{r}{n})}$

$\lim_{n\to\infty}n\ln(1+\frac{r}{n})=\lim_{n\to\infty}\dfrac{\ln(1+\frac{r}{n})}{\frac{1}{n}}$

$$=\lim_{n\to\infty}\frac{\left(\frac{-\frac{r}{n^2}}{1+\frac{r}{n}}\right)}{-\frac{1}{n^2}}=\lim_{n\to\infty}\frac{r}{1+\frac{r}{n}}=r$$

$A_\infty(r)=Ae^r$