Methods in Calculus 7A

1 a i $y = e^{5x}, \frac{dy}{dx} = 5e^{5x}, \frac{d^2y}{dx^2} = 5^2e^{5x} = 25e^{5x}$

$\frac{d^3y}{dx^3} = 5^3e^{5x} = 125e^{5x}$

ii $\frac{d^ny}{dx^n} = 5^n e^{5x}$

b i $y = e^{-x}, \frac{dy}{dx} = -e^{-x}, \frac{d^2y}{dx^2} = (-1)^2e^{-x} = e^{-x}$

$\frac{d^3y}{dx^3} = (-1)^3e^{-x} = -e^{-x}$

ii $\frac{d^ny}{dx^n} = (-1)^n e^{-x}$

c i $y = x^m, \frac{dy}{dx} = mx^{m-1}, \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$

$\frac{d^3y}{dx^3} = m(m-1)(m-2)x^{m-3}$

ii

$\frac{d^ny}{dx^n} = m(m-1)(m-2)...(m-(n-1))x^{n-m}$

$= \frac{m!}{(m-n)!}x^{m-n} \quad (m \geq n)$

d i $y = xe^{-x}, u = x, v = e^{-x}$

$\frac{du}{dx} = 1, \frac{d^2u}{dx^2} = 0 = \frac{d^nu}{dx^n}, n > 2$

$\frac{dv}{dx} = -e^{-x}, \frac{d^2v}{dx^2} = e^{-x}$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = -xe^{-x} + e^{-x} = (1-x)e^{-x}$

$\frac{d^2y}{dx^2} = -\frac{dy}{dx} - e^{-x} = xe^{-x} - 2e^{-x} = (x-2)e^{-x}$

$\frac{d^3y}{dx^3} = -xe^{-x} + 3e^{-x} = (3-x)e^{-x}$

ii $\frac{d^ny}{dx^n} = (-1)^n(y - ne^{-x}) = (-1)^n(x-n)e^{-x}$

2 a $y = (2x^2 + x - 2)(4x^2 - 3x + 8)$

$u = 2x^2 + x - 2, \frac{du}{dx} = 4x + 1, \frac{d^2u}{dx^2} = 4$

$v = 4x^2 - 3x + 8, \frac{dv}{dx} = 8x - 3, \frac{d^2v}{dx^2} = 8$

$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$

$= 8(2x^2 + x - 2) + 2(4x + 1)(8x - 3) + 4(4x^2 - 3x + 8)$

$= 96x^2 - 12x + 10$

b $y = \ln x \sin x$

$u = \ln x, \frac{du}{dx} = \frac{1}{x}, \frac{d^2u}{dx^2} = -\frac{1}{x^2}$

$v = \sin x, \frac{dv}{dx} = \cos x, \frac{d^2v}{dx^2} = -\sin x$

$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$

$= -\ln x \sin x + \frac{2\cos x}{x} - \frac{\sin x}{x^2}$

c

$y = e^{3x}\cos 2x$

$u = e^{3x}, \frac{du}{dx} = 3e^{3x}, \frac{d^2u}{dx^2} = 9e^{3x}$

$v = \cos 2x, \frac{dv}{dx} = -2\sin 2x, \frac{d^2v}{dx^2} = -4\cos 2x$

$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$

$= -4e^{3x}\cos 2x + 2(3e^{3x})(-2\sin 2x) + 9e^{3x}\cos 2x$

$= 5e^{3x}\cos 2x - 12e^{3x}\sin 2x$

2 d $y = x^3 \ln(2x+1)$

$u = x^3, \frac{du}{dx} = 3x^2, \frac{d^2u}{dx^2} = 6x$

$v = \ln(2x+1), \frac{dv}{dx} = \frac{2}{2x+1}$

$\frac{d^2v}{dx^2} = \frac{-4}{(2x+1)^2}$

$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$

$= 6x\ln(2x+1) + \frac{12x^2}{2x+1} - \frac{4x^3}{(2x+1)^2}$

e

$y = (x^2 - x + 2)(x^3 - 1)$

$u = x^2 - x + 2, \frac{du}{dx} = 2x - 1, \frac{d^2u}{dx^2} = 2, \frac{d^3u}{dx^3} = 0$

$v = x^3 - 1, \frac{dv}{dx} = 3x^2, \frac{d^2v}{dx^2} = 6x, \frac{d^3v}{dx^3} = 6$

$\frac{d^3y}{dx^3} = u\frac{d^3v}{dx^3} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + v\frac{d^3u}{dx^3}$

$= 6(x^2 - x + 2) + 3(2x-1)(6x) + 3(2)(3x^2)$

$= 60x^2 - 24x + 12$

f $y = \sqrt{2x}\sinh 3x$

$u = \sqrt{2x}, \frac{du}{dx} = (2x)^{-\frac{1}{2}}, \frac{d^2u}{dx^2} = -(2x)^{-\frac{3}{2}}$

$v = \sinh 3x, \frac{dv}{dx} = 3\cosh 3x, \frac{d^2v}{dx^2} = 9\sinh 3x$

$\frac{d^3v}{dx^3} = 27\cosh 3x, \frac{d^3u}{dx^3} = 3(2x)^{-\frac{5}{2}}$

$\frac{d^3y}{dx^3} = u\frac{d^3v}{dx^3} + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + v\frac{d^3u}{dx^3}$

$= 27\sqrt{2x}\cosh 3x + \frac{27}{\sqrt{2x}}\sinh 3x$

$-\frac{9}{2x\sqrt{2x}}\cosh 3x + \frac{3}{4x^2\sqrt{2x}}\sinh 3x$

g $y = (x^2 - x)\cosh 2x$

$u = x^2 - x, \frac{du}{dx} = 2x, \frac{d^2u}{dx^2} = 2$

$\frac{d^3u}{dx^3} = \frac{d^4u}{dx^4} = 0$

$v = \cosh 2x, \frac{dv}{dx} = 2\sinh 2x, \frac{d^2v}{dx^2} = 4\cosh 2x$

$\frac{d^3v}{dx^3} = 8\sinh 2x, \frac{d^4v}{dx^4} = 16\cosh 2x$

$\frac{d^4y}{dx^4} = u\frac{d^4v}{dx^4} + 4\frac{d^3u}{dx^3}\frac{dv}{dx}$

$+6\frac{d^2u}{dx^2}\frac{d^2v}{dx^2} + 4\frac{du}{dx}\frac{d^3v}{dx^3} + v\frac{d^4u}{dx^4}$

$= 16(x^2 - x)\cosh 2x + 4(2x-1)(8\sinh 2x)$

$+6(2)(4\cosh 2x)$

$= 16(x^2 - x + 3)\cosh 2x + 32(2x-1)\sinh 2x$

h $y = \cos x \sinh x$

$u = \cos x, \frac{du}{dx} = -\sin x, \frac{d^2u}{dx^2} = -\cos x$

$\frac{d^3u}{dx^3} = \sin x, \frac{d^4u}{dx^4} = \cos x$

$v = \sinh x, \frac{dv}{dx} = \cosh x, \frac{d^2v}{dx^2} = \sinh x$

$\frac{d^3v}{dx^3} = \cosh x, \frac{d^4v}{dx^4} = \sinh x$

$\frac{d^3y}{dx^3} = u\frac{d^4v}{dx^4} + 4\frac{d^3u}{dx^3}\frac{dv}{dx}$

$+6\frac{d^2u}{dx^2}\frac{d^2v}{dx^2} + 4\frac{du}{dx}\frac{d^3v}{dx^3} + v\frac{d^4u}{dx^4}$

$= \cos x\sinh x - 4\sin x\cosh x - 6\cos x\sinh x$

$+4\sin x\cosh x + \cos x\sinh x$

$= -4\cos x\sinh x$

3 a $y = \dfrac{\sqrt{x}}{\ln x}$

$u = \sqrt{x}, \dfrac{du}{dx} = \dfrac{1}{2}(x)^{-\frac{1}{2}}, \dfrac{d^2u}{dx^2} = -\dfrac{1}{4}(x)^{-\frac{3}{2}}$

$v = (\ln x)^{-1}, \dfrac{dv}{dx} = -\dfrac{1}{x}(\ln x)^{-2}$

$\dfrac{d^2v}{dx^2} = \dfrac{1}{x^2}(\ln x)^{-2} + \dfrac{2}{x^2}(\ln x)^{-3}$

$\dfrac{d^2y}{dx^2} = u\dfrac{d^2v}{dx^2} + 2\dfrac{du}{dx}\dfrac{dv}{dx} + v\dfrac{d^2u}{dx^2}$

$= \dfrac{1}{x\sqrt{x}}(\ln x)^{-2} + \dfrac{2}{x\sqrt{x}}(\ln x)^{-3}$

$-2\dfrac{1}{2x\sqrt{x}}(\ln x)^{-2} - \dfrac{1}{4x\sqrt{x}}(\ln x)^{-1}$

$= \dfrac{8 - (\ln x)^2}{4x\sqrt{x}(\ln x)^3}$

b $y = \dfrac{\ln x}{x+3}$

$u = \dfrac{1}{x+3}, \dfrac{du}{dx} = \dfrac{-1}{(x+3)^2}$

$\dfrac{d^2u}{dx^2} = \dfrac{2}{(x+3)^3}, \dfrac{d^3u}{dx^3} = \dfrac{-6}{(x+3)^4}$

$v = \ln x, \dfrac{dv}{dx} = \dfrac{1}{x}, \dfrac{d^2v}{dx^2} = -\dfrac{1}{x^2}, \dfrac{d^3v}{dx^3} = \dfrac{2}{x^3}$

$\dfrac{d^3y}{dx^3} = u\dfrac{d^3v}{dx^3} + 3\dfrac{d^2u}{dx^2}\dfrac{dv}{dx} + 3\dfrac{du}{dx}\dfrac{d^2v}{dx^2} + v\dfrac{d^3u}{dx^3}$

$= \dfrac{2}{x^3(x+3)} + \dfrac{3}{x^2(x+3)^2} + \dfrac{6}{x(x+3)^3} - \dfrac{6\ln x}{(x+3)^4}$

$= \dfrac{2(x+3)^3 + 3x(x+3)^2 + 6x^2(x+3) - 6x^3\ln x}{x^3(x+3)^4}$

$= \dfrac{11x^3 - 6x^3\ln x + 54x^2 + 81x + 54}{x^3(x+3)^4}$

c $y = \dfrac{e^x+1}{e^x-1}$

$u = e^x + 1, \dfrac{du}{dx} = e^x$

$v = \dfrac{1}{e^x-1}, \dfrac{dv}{dx} = \dfrac{-e^x}{(e^x-1)^2}$

$\dfrac{dy}{dx} = \dfrac{-e^x}{(e^x-1)^2}(e^x+1) + \dfrac{e^x}{e^x-1} = \dfrac{-2e^x}{(e^x-1)^2}$

$u = \dfrac{du}{dx} = -2e^x$

$v = \dfrac{1}{(e^x-1)^2}, \dfrac{dv}{dx} = \dfrac{-2e^x}{(e^x-1)^3}$

$\dfrac{d^2y}{dx^2} = \dfrac{4e^{2x}}{(e^x-1)^3} - \dfrac{2e^x}{(e^x-1)^2} = \dfrac{2e^x(e^x+1)}{(e^x-1)^3}$

$u = 2e^x(e^x+1), \dfrac{du}{dx} = 4e^{2x} + 2e^x$

$v = \dfrac{1}{(e^x-1)^3}, \dfrac{dv}{dx} = \dfrac{-3e^x}{(e^x-1)^4}$

$\dfrac{d^3y}{dx^3} = \dfrac{-6e^{2x}(e^x+1)}{(e^x-1)^4} + \dfrac{4e^{2x}+2e^x}{(e^x-1)^3}$

$= \dfrac{-2e^x(e^{2x}+4e^x+1)}{(e^x-1)^4}$

d $y = \dfrac{1}{4x^2}\sin x$

$u = \dfrac{1}{4x^2}, \dfrac{du}{dx} = -\dfrac{1}{2x^3}, \dfrac{d^2u}{dx^2} = \dfrac{3}{2x^4}$

$\dfrac{d^3u}{dx^3} = -6x^{-5}, \dfrac{d^4u}{dx^4} = 30x^{-6}$

$v = \sin x, \dfrac{dv}{dx} = \cos x, \dfrac{d^2v}{dx^2} = -\sin x$

$\dfrac{d^3v}{dx^3} = -\cos x, \dfrac{d^4v}{dx^4} = \sin x$

$\dfrac{d^4y}{dx^4} = u\dfrac{d^4v}{dx^4} + 4\dfrac{d^3u}{dx^3}\dfrac{dv}{dx}$

$+6\dfrac{d^2u}{dx^2}\dfrac{d^2v}{dx^2} + 4\dfrac{du}{dx}\dfrac{d^3v}{dx^3} + v\dfrac{d^4u}{dx^4}$

$= \left(\dfrac{1}{4x^2} - \dfrac{9}{x^4} + \dfrac{30}{x^6}\right)\sin x + \left(\dfrac{2}{x^3} - \dfrac{24}{x^5}\right)\cos x$

4 $y = e^x \cos x$

$u = e^x = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}^n u}{\mathrm{d}x^n}$

$v = \cos x, \frac{\mathrm{d}v}{\mathrm{d}x} = -\sin x$

$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \frac{\mathrm{d}^6 v}{\mathrm{d}x^6} = -v$

$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}^5 v}{\mathrm{d}x^3} = -\frac{\mathrm{d}^3 v}{\mathrm{d}x^3}$

$\frac{\mathrm{d}^6 y}{\mathrm{d}x^6} + 8\frac{\mathrm{d}y}{\mathrm{d}x} = e^x(\cos x - 6\sin x - 15\cos x$

$20\sin x + 15\cos x - 6\sin x - \cos x)$

$+8e^x(\cos x - \sin x)$

$= 8e^x \cos x = 8y$

Hence $8e^x \cos x - 8y = 0$

5 $y = 2x^3 e^{2x}$

$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{\mathrm{d}^k u}{\mathrm{d}x^k} \frac{\mathrm{d}^{n-k} v}{\mathrm{d}x^{n-k}}$

$u = 2x^3, \frac{\mathrm{d}u}{\mathrm{d}x} = 6x^2, \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 12x, \frac{\mathrm{d}^3 u}{\mathrm{d}x^3} = 12$

$\frac{\mathrm{d}^k u}{\mathrm{d}x^k} = 0, k > 3$

$v = e^{2x}, \frac{\mathrm{d}^{n-k} v}{\mathrm{d}x^{n-k}} = 2^{n-k} e^{2x}$

$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 2x^3 2^n e^{2x} + 6\binom{n}{1} x^2 2^{n-1} e^{2x}$

$+12\binom{n}{2} x 2^{n-2} e^{2x} + 12\binom{n}{3} 2^{n-3} e^{2x}$

$= 2^{n-2} e^{2x}\left(8x^3 + 12nx^2 + 12\binom{n}{2}x + 6\binom{n}{3}\right)$

$= 2^{n-2} e^{2x}\left(8x^3 + 12nx^2 + 6n(n-1)x + n(n-1)(n-2)\right)$

6 a $y = \frac{1}{x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}$

Assume $\frac{\mathrm{d}^k y}{\mathrm{d}x^k} = (-1)^k \frac{k!}{x^{k+1}}$

Then $\frac{\mathrm{d}^{k+1} y}{\mathrm{d}x^{k+1}} = (-1)^k k! \frac{-(k+1)}{x^{k+2}}$

$= (-1)^{k+1} \frac{(k+1)!}{x^{k+2}}$

So true for $n = k+1$ if true for $n = k$,

true for $n = 1$ so true for $\forall$ natural numbers n

b From part **a**,

$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{In}x) = \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left(\frac{1}{x}\right)$

$= (-1)^{n-1} \frac{(n-1)!}{x^n}$ if $n \geqslant 1$

Hence by Leibnitz's theorem

$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^3 \mathrm{In}x\right) = \sum_{k=0}^{n} \binom{n}{k} \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^3\right) \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(x^3\right) \frac{\mathrm{d}^{n-k}}{\mathrm{d}x^{n-k}}(\mathrm{In}x)$

$= (-1)^{n-1} \frac{(n-1)!}{x^{n-3}} + 3n(-1)^{n-2} \frac{(n-2)!}{x^{n-3}}$

$+ \frac{n(n-1)}{2} 6(-1)^{n-3} \frac{(n-3)!}{x^{n-3}}$

$+ \frac{n(n-1)(n-2)}{6} 6(-1)^{n-4} \frac{(n-4)!}{x^{n-3}}$

$= \frac{(-1)^n (n-4)!}{x^{n-3}}(-(n-1)(n-2)(n-3)$

$+3n(n-2)(n-3) - 3n(n-1)(n-3)$

$+n(n-1)(n-2))$

$= \frac{6(-1)^n (n-4)!}{x^{n-3}}$

7 $y = x^2 \sinh kx$

$u = x^2, \frac{du}{dx} = 2x, \frac{d^2u}{dx^2} = 2, \frac{d^3u}{dx^3} = 0$

$v = \sinh kx$

n even, $k > 2$

$\frac{d^n v}{dx^n} = k^n \sinh kx, \frac{d^{n-1}v}{dx^{n-1}} = k^{n-1}\cosh kx$

$\frac{d^{n-2}v}{dx^{n-2}} = k^{n-2}\sinh kx$

$\frac{d^n y}{dx^n} = x^2 k^n \sinh kx + 2nxk^{n-1}\cosh kx$

$+2\binom{n}{2}k^{n-2}\sinh kx$

$= k^{n-2}\sinh kx(x^2k^2 + n(n-1)) + 2nxk^{n-1}\cosh kx$

n odd, $k > 2$

$\frac{d^n v}{dx^n} = k^n \cosh kx, \frac{d^{n-1}v}{dx^{n-1}} = k^{n-1}\sinh kx$

$\frac{d^{n-2}v}{dx^{n-2}} = k^{n-2}\cosh kx$

$\frac{d^n y}{dx^n} = x^2 k^n \cosh kx + 2nxk^{n-1}\sinh kx$

$+2\binom{n}{2}k^{n-2}\cosh kx$

$= k^{n-2}\cosh kx(x^2k^2 + n(n-1)) + 2nxk^{n-1}\sinh kx$

check $n = 1: \frac{dy}{dx} = kx^2\cosh kx + 2x\sinh kx$

$n = 2: \frac{dy}{dx} = k^2x^2\sinh kx + 2\sinh kx + 4x\cosh kx$

also satisfy the required formulae.

Challenge

a $F(x) = f(x)g(x)$

$F^{(n)}(x) = \sum_{k=0}^{n}\binom{n}{k}f^{(k)}(x)g^{(n-k)}(x), n = 1$

$F'(x) = \binom{1}{0}f^{(0)}(x)g^{(1)}(x) + \binom{1}{1}f^{(1)}(x)g^{(0)}(x)$

$= f(x)g'(x) + f'(x)g(x)$

Challenge

b $F^{(m)}(x) = \sum_{k=0}^{m}\binom{m}{k}f^{(k)}(x)g^{(m-k)}(x)$

Assume true for $m = n$, then

$F^{(n+1)}(x) = \frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(x)g^{(n-k)}(x)$

$= \sum_{k=0}^{n}\binom{n}{k}\frac{d}{dx}f^{(k)}(x)g^{(n-k)}(x)$

$= \sum_{k=0}^{n}\binom{n}{k}\left[f^{(k)}(x)g^{(n-k+1)}(x) + f^{(k+1)}(x)g^{(n-k)}(x)\right]$

$= \sum_{k=0}^{n}\binom{n}{k}f^{(k)}(x)g^{(n-k+1)}(x)$

$+ \sum_{k=1}^{n+1}\binom{n}{k-1}f^{((k-1)+1)}(x)g^{(n-(k-1))}(x)$

$= \binom{n}{0}f(x)g^{(n+1)}(x) + \binom{n}{n}f^{(n+1)}(x)g(x)$

$+ \sum_{k=1}^{n}\left(\binom{n}{k} + \binom{n}{k-1}\right)f^{(k)}(x)g^{(n-k+1)}(x)$

$= \sum_{k=0}^{n+1}\binom{n+1}{k}f^{(k)}(x)g^{(n+1-k)}(x)$

Using the problem-solving hint

True for $n = k+1$ if true for $n = k$,

true for $n = 1$ (challenge **a**)

so true for $\forall$ natural numbers n