Taylor Series Mixed Exercise 6

1 Let $\mathrm{f}(x) = \left(x - \frac{\pi}{4}\right)\cot x$ and $a = \frac{\pi}{4} \Rightarrow \mathrm{f}(a) = 0$

$$\mathrm{f}'(x) = \left(x - \frac{\pi}{4}\right)\left(-\mathrm{cosec}^2 x\right) + \cot x \Rightarrow \mathrm{f}'(a) = 1$$

$$\mathrm{f}''(x) = \left(x - \frac{\pi}{4}\right)2\cot x\,\mathrm{cosec}^2 x + \left(-2\,\mathrm{cosec}^2 x\right) \Rightarrow \mathrm{f}''(a) = -4$$

$$\mathrm{f}'''(x) = \left(x - \frac{\pi}{4}\right)\left(-2\,\mathrm{cosec}^4 x - 4\cot^2 x\,\mathrm{cosec}^2 x\right) + 6\cot x\,\mathrm{cosec}^2 x \Rightarrow \mathrm{f}'''(a) = 12$$

Substituting into the Taylor series expansion gives

$$\mathrm{f}(x) = 0 + 1\left(x - \frac{\pi}{4}\right) + \frac{-4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{12}{3!}\left(x - \frac{\pi}{4}\right)^3 + \ldots$$

$$= \left(x - \frac{\pi}{4}\right) - 2\left(x - \frac{\pi}{4}\right)^2 + 2\left(x - \frac{\pi}{4}\right)^3 + \ldots \text{ as required}$$

2 **a** $\mathrm{f}(x) = \ln(1 + \mathrm{e}^x)$ so $\mathrm{f}(0) = \ln 2$

$\mathrm{f}'(x) = \frac{\mathrm{e}^x}{1 + \mathrm{e}^x}$ $\quad \mathrm{f}'(0) = \frac{1}{2}$

$= 1 - \frac{1}{1 + \mathrm{e}^x} = 1 - (1 + \mathrm{e}^x)^{-1}$

So $\mathrm{f}''(x) = \frac{\mathrm{e}^x}{(1 + \mathrm{e}^x)^2}$ or use the quotient rule $\quad \mathrm{f}''(0) = \frac{1}{4}$

b $\mathrm{f}'''(x) = \frac{(1 + \mathrm{e}^x)^2\mathrm{e}^x - \mathrm{e}^x 2(1 + \mathrm{e}^x)\mathrm{e}^x}{(1 + \mathrm{e}^x)^4}$

Use the quotient rule and chain rule.

$$= \frac{(1 + \mathrm{e}^x)\mathrm{e}^x\{(1 + \mathrm{e}^x) - 2\mathrm{e}^x\}}{(1 + \mathrm{e}^x)^4} = \frac{\mathrm{e}^x(1 - \mathrm{e}^x)}{(1 + \mathrm{e}^x)^3}$$

$\mathrm{f}'''(0) = 0$

c Using Maclaurin's expansion:

$$\ln(1 + \mathrm{e}^x) = \ln 2 + \frac{x}{2} + \frac{x^2}{8} + \ldots$$

The expansion is valid for $-1 < \mathrm{e}^x \leqslant 1 \Rightarrow 0, \mathrm{e}^x \leqslant 1$ so for $x \leqslant 0$

3 **a** $\cos 4x = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} - \frac{(4x)^6}{6!} + \ldots$

$$= 1 - 8x^2 + \frac{32}{3}x^4 - \frac{256}{45}x^6 + \ldots$$

b $\cos 4x = 1 - 2\sin^2 2x$,

so $2\sin^2 2x = 1 - \cos 4x = 8x^2 - \frac{32}{3}x^4 + \frac{256}{45}x^6 + \ldots$

$$\sin^2 2x = 4x^2 - \frac{16}{3}x^4 + \frac{128}{45}x^6 + \ldots$$

4 Using $\mathrm{e}^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$ and $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots$

$$\mathrm{e}^{\cos x} = \mathrm{e}^{\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)} = \mathrm{e} \times \mathrm{e}^{-\frac{x^2}{2}} \times \mathrm{e}^{\frac{x^4}{24}}$$

$$= \mathrm{e}\left\{1 + \left(-\frac{x^2}{2}\right) + \frac{1}{2}\left(-\frac{x^2}{2}\right)^2 + \ldots\right\}\left\{1 + \frac{x^4}{24} + \ldots\right\} \quad \text{no other terms required}$$

$$= \mathrm{e}\left\{1 - \frac{x^2}{2} + \frac{x^4}{8} + \ldots\right\}\left\{1 + \frac{x^4}{24} + \ldots\right\}$$

$$= \mathrm{e}\left\{1 - \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^4}{24} + \ldots\right\} = \mathrm{e}\left\{1 - \frac{x^2}{2} + \frac{x^4}{6} + \ldots\right\}$$

5 a $\frac{\mathrm{d}y}{\mathrm{d}x} = 2 + x + \sin y$ and $x_0 = 0,\ y_0 = 0$ **(1)** so $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 2$

Differentiating **(1)** gives $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 1 + \cos y \frac{\mathrm{d}y}{\mathrm{d}x}$ **(2)**

Substituting $x_0 = 0,\ y_0 = 0,\ \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 2$ into **(2)** gives $\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_0 = 3$

Differentiating **(2)** gives $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = \cos y \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \sin y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ **(3)**

Substituting $y_0 = 0,\ \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 2,\ \left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_0 = 3$ into **(3)** gives $\left(\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}\right)_0 = 3$

Substituting found values into $y = y_0 + x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 + \frac{x^2}{2!}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_0 + \frac{x^3}{3!}\left(\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}\right)_0 + \ldots$

$$y = 2x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \ldots$$

b At $x = 0.1,\ y \approx 2(0.1) + \frac{3}{2}(0.1)^2 + \frac{1}{2}(0.1)^3 = 0.2155$

6 $\ln[(1+x)^2(1-2x)] = 2\ln(1+x) + \ln(1-2x)$

$$= 2\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right\} + \left\{(-2x) - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \ldots\right\}$$

$$= 2x - x^2 + \frac{2}{3}x^3 - \frac{1}{2}x^4 - 2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 + \ldots$$

$$= -3x^2 - 2x^3 - \ldots$$

7 $\dfrac{d^2y}{dx^2}-(x+2)\dfrac{dy}{dx}+3y=0$ **(1)**

Differentiating **(1)** gives $\dfrac{d^3y}{dx^2}-(x+2)\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}+3\dfrac{dy}{dx}=0$

So that $\dfrac{d^3y}{dx^2}-(x+2)\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}=0$ **(2)**

Substituting initial data in **(1)** gives $\left(\dfrac{d^2y}{dx^2}\right)_0=2$

Substituting known data in **(2)** gives $\left(\dfrac{d^3y}{dx^3}\right)_0=-4$

So $y=2+4x+\dfrac{2x^2}{2!}-\dfrac{4x^3}{3!}+\ldots$

$=2+4x+x^2-\dfrac{2}{3}x^3$

8 a $f(x)=\ln(\sec x+\tan x)$ $f(0)=\ln 1=0$

$f'(x)=\dfrac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}=\dfrac{\sec x(\tan x+\sec x)}{\sec x+\tan x}=\sec x$ $f'(0)=1$

$f''(x)=\sec x\tan x$ $f''(0)=0$

$f'''(x)=\sec x\sec^2 x+\sec x\tan x\tan x$ $f'''(0)=1$

Substituting into Maclaurin's expansion gives $y=x+\dfrac{x^3}{6}+\ldots$

b We use the expansions:

$\ln(\sec x+\tan x)=x+\dfrac{1}{6}x^3+\ldots$

$\sin x=x-\dfrac{1}{3!}x^3+\ldots$

to see that:

$\dfrac{\sin x-\ln(\sec x+\tan x)}{x(\cos x-1)}$

$=\dfrac{\left(x-\frac{1}{3!}x^3+\ldots\right)-\left(x+\frac{1}{6}x^3+\ldots\right)}{x(1-\frac{1}{2}x^2+\ldots-1)}$

$=\dfrac{-\frac{1}{3}x^3+\ldots}{-\frac{1}{2}x^3+\ldots}$

$\Rightarrow \lim\limits_{x\to 0}\dfrac{\sin x-\ln(\sec x+\tan x)}{x(\cos x-1)}$

$=\lim\limits_{x\to 0}\dfrac{-\frac{1}{3}x^3+\ldots}{-\frac{1}{2}x^3+\ldots}=\dfrac{2}{3}$

9 We first make note of the fact that:

$$\sinh(\ln 2) = \frac{1}{2}(e^{\ln 2} - e^{-\ln 2}) = \frac{1}{2}\left(2 - \frac{1}{2}\right) = \frac{3}{4}$$

$$\cosh(\ln 2) = \frac{1}{2}(e^{\ln 2} + e^{-\ln 2}) = \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{5}{4}$$

and that $\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$, $\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x$, which implies that:

$$\frac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}\cosh x = \cosh x, \frac{\mathrm{d}^{2k+1}}{\mathrm{d}x^{2k+1}}\cosh x = \sinh x$$

$$\Rightarrow \left(\frac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}\cosh x\right)_{x=\ln 2} = \frac{5}{4}, \left(\frac{\mathrm{d}^{2k+1}}{\mathrm{d}x^{2k+1}}\cosh x\right)_{x=\ln 2} = \frac{3}{4}$$

Then, the Taylor series about $x = \ln 2$ is:

$$\cosh x = \sum_{l=0}^{\infty} \frac{1}{l!}(x - \ln 2)^l \left(\frac{\mathrm{d}^l}{\mathrm{d}x^l}\cosh x\right)_{x=\ln 2}$$

Thus we deduce that:

a The n^{th} term when n is even is:

$$\frac{5}{4n!}(x - \ln 2)^n$$

b The n^{th} term when n is odd is:

$$\frac{3}{4n!}(x - \ln 2)^n$$

10 Consider the first two terms in the Taylor series of $\cos x$ around $x = \pi$:

$$\cos x = -1 + \frac{1}{2}(x - \pi)^2 + \frac{1}{24}(x - \pi)^4 + \ldots$$

Then the limit is given by:

$$\lim_{x\to\pi} \frac{(x-\pi)^2}{1+\cos x} = \lim_{x\to\pi} \frac{(x-\pi)^2}{\frac{1}{2}(x-\pi)^2 + \frac{1}{24}(x-\pi)^4 + \ldots}$$

$$= \lim_{x\to\pi} \frac{1}{\frac{1}{2} + \frac{1}{24}(x-\pi)^2 + \ldots} = 2$$

11 Consider the first two terms in the Taylor series:

$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots$

$\Rightarrow \arctan x - x = -\frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots$

$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots$

$\Rightarrow \sin x - x = -\frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots$

Then we can evaluate the limit:

$$\lim_{x\to 0}\frac{\arctan x - x}{\sin x - x} = \lim_{x\to 0}\frac{-\frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots}{-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots}$$

$$= \lim_{x\to 0}\frac{-\frac{1}{3} + \frac{1}{5}x^2 + \ldots}{-\frac{1}{6} + \frac{1}{5!}x^2 + \ldots} = 2$$

12 a We differentiate the respective Taylor series term by term and match that up with the derivative. Firstly:

$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{r!}x^r + \cdots$

$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}e^x = 1 + \frac{2}{2!}x + \frac{3}{3!}x^2 + \frac{4}{4!}x^3 + \cdots$

$+ \frac{r}{r!}x^{r-1} + \frac{r+1}{(r+1)!}x^r + \cdots$

$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$

$+ \frac{1}{(r-1)!}x^{r-1} + \frac{1}{r!}x^r + \cdots = e^x$

b $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = 1 - \frac{3}{3!}x^2 + \cdots + \frac{(-1)^r(2r+1)}{(2r+1)!}x^{2r} + \cdots$

$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\sin x = 1 - \frac{1}{2!}x^2 + \cdots + \frac{(-1)^r}{(2r)!}x^{2r} + \cdots = \cos x$

c $\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\frac{2}{2!}x + \frac{4}{4!}x^3 + \cdots + \frac{(-1)^r(2r)}{(2r)!}x^{2(r-1)+1} + \cdots$

$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -x + \frac{1}{3!}x^3 + \cdots + \frac{(-1)^r}{(2r-1)!}x^{2(r-1)+1} + \cdots$

$= -\left(x - \frac{1}{3!}x^3 + \cdots + \frac{(-1)^{r-1}}{(2r-1)!}x^{2(r-1)+1} + \cdots\right) = -\sin x$

13 $\dfrac{d^2y}{dx^2} + y\dfrac{dy}{dx} = x$ **(1)**

Differentiating $\dfrac{d^2y}{dx^2} + y\dfrac{dy}{dx} = x$, gives $\dfrac{d^3y}{dx^3} + y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 1$ **(2)**

Substituting initial values into **(1)** gives $\left(\dfrac{d^2y}{dx^2}\right)_1 = 1$

Substituting $\left(\dfrac{dy}{dx}\right)_1 = 2$ and $\left(\dfrac{d^2y}{dx^2}\right)_1 = 1$ into **(2)** gives $\left(\dfrac{d^3y}{dx^3}\right) = -3$.

Using Taylor's expansion in the form with $x_0 = 1$

$$y = 0 + 2(x-1) + \frac{(1)}{2!}(x-1)^2 + \frac{(-3)}{3!}(x-1)^3 + \dots$$

$$= 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{2}(x-1)^3 + \dots$$

14 a You can write $\cos x = 1 - \left(\dfrac{x^2}{2} - \dfrac{x^4}{24} + \dots\right)$; it is not necessary to have higher powers:

$$\sec x = \frac{1}{\cos x} = \frac{1}{1-\left(\frac{x^2}{2} - \frac{x^4}{24} + \dots\right)} = \left\{1 - \left(\frac{x^2}{2} - \frac{x^4}{24} + \dots\right)\right\}^{-1}$$

Using the binomial expansion but only requiring powers up to x^4

$$\sec x = 1 + (-1)\left\{-\left(\frac{x^2}{2} - \frac{x^4}{24}\right)\right\} + \frac{(-1)(-2)}{2!}\left\{-\left(\frac{x^2}{2} - \frac{x^4}{24}\right)\right\}^2 + \dots$$

$$= 1 + \left(\frac{x^2}{2} - \frac{x^4}{24}\right) + \frac{x^4}{4} + \text{higher powers of } x$$

$$= 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \dots$$

b $\tan x = \dfrac{\sin x}{\cos x} = \sin x \times \sec x$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)\left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \dots\right)$$

$$= x + \frac{x^3}{2} + \frac{5}{24}x^5 - \frac{x^3}{3!} - \frac{1}{2(3!)}x^5 + \frac{x^5}{5!} + \dots$$

$$= x + \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{5}{24} - \frac{1}{12} + \frac{1}{120}\right)x^5 + \dots$$

$$= x + \frac{x^3}{3} + \frac{16}{120}x^5 + \dots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots$$

14 c Using the series expansions:

$$\tan x = x + \frac{x^3}{3} + \ldots$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \ldots \Rightarrow e^{2x} - 1 = 2x + \frac{4x^2}{2!} + \ldots$$

we can evalaute the limit:

$$\lim_{x\to 0} \frac{\tan x}{e^{2x} - 1} = \lim_{x\to 0} \frac{x + \frac{x^3}{3} + \ldots}{2x + \frac{4x^2}{2!} + \ldots} = \lim_{x\to 0} \frac{1 + \frac{x^2}{3} + \ldots}{2 + \frac{4x}{2!} + \ldots} = \frac{1}{2}$$

15 a Using $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ and $\cos 3x = 1 - \frac{(3x)^2}{2!} + \ldots$

$$\mathrm{e}^x \cos 3x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right)\left(1 - \frac{9x^2}{2} + \ldots\right)$$

$$= \left\{1 + x + \left(\frac{x^2}{2} - \frac{9x^2}{2}\right) + \left(\frac{x^3}{6} - \frac{9x^3}{2}\right) + \ldots\right\}$$

$$= 1 + x - 4x^2 - \frac{13}{3}x^3 + \ldots$$

b Using the series expansion in the first part we can deduce that:

$$e^x \cos 3x = 1 + x - 4x^2 - \frac{13}{3}x^3 + \ldots$$

$$\sin x = x - \frac{1}{6}x^3 + \ldots$$

$$\cos x = 1 - \frac{1}{2}x^2 + \ldots$$

$$\Rightarrow e^x \cos 3x - \sin x - \cos x = -\frac{7}{2}x^2 - \frac{25}{6}x^3 + \ldots$$

Then we can evaulate the limit:

$$\lim_{x\to 0} \frac{e^x \cos 3x - \sin x - \cos x}{x^3 + 2x^2}$$

$$= \lim_{x\to 0} \frac{-\frac{7}{2}x^2 - \frac{25}{6}x^3 + \ldots}{2x^2 + x^3} = \lim_{x\to 0} \frac{-\frac{7}{2} - \frac{25}{6}x + \ldots}{2 + x} = -\frac{7}{4}$$

16 a Differentiating $\frac{d^2y}{dx^2}+x^2\frac{dy}{dx}+y=0$ **(1)** with respect to x, gives:

$$\frac{d^3y}{dx^3}+2x\frac{dy}{dx}+x^2\frac{d^2y}{dx^2}+\frac{dy}{dx}=0 \qquad \textbf{(2)}$$

Substituting given data $x_0=0,\ y_0=2$ and $\left(\frac{dy}{dx}\right)_0=1$ into **(1)** gives $\left(\frac{d^2y}{dx^2}\right)_0=-2$

Substituting $x_0=0,\ \left(\frac{dy}{dx}\right)_0=1$ and $\left(\frac{d^2y}{dx^2}\right)_0=-2$ into **(2)** gives $\left(\frac{d^3y}{dx^3}\right)_0=-1$

So using Taylor series $y=y_0+x\left(\frac{dy}{dx}\right)_0+\frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0+\frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0+\ldots$

$$y=2+x-x^2-\frac{x^3}{6}+\ldots$$

b Differentiating **(2)** with respect to x gives:

$$\frac{d^4y}{dx^4}+2x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+x^2\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}+\frac{d^2y}{dx^2}=0 \qquad \textbf{(3)}$$

Substituting $x=0,\ \left(\frac{dy}{dx}\right)_0=1,\ \left(\frac{d^2y}{dx^2}\right)_0=-2$ and $\left(\frac{d^3y}{dx^3}\right)_0=-1$ into **(3)** gives,

at $x=0,\ \frac{d^4y}{dx^4}+2(1)+(-2)=0$, so $\frac{d^4y}{dx^4}=0$

17 a $f(x)=(1+x)^2\ln(1+x)$

$f'(x)=(1+x)^2\frac{1}{1+x}+2(1+x)\ln(1+x)=(1+x)(1+2\ln(1+x))$

$f''(x)=(1+x)\left(\frac{2}{1+x}\right)+(1+2\ln(1+x))=3+2\ln(1+x)$

$f'''(x)=\left(\frac{2}{1+x}\right)$

$f(0)=0, f'(0)=1,\ f''(0)=3,\ f'''(0)=2$

b Using Maclaurin's expansion

$$(1+x)^2\ln(1+x)=0+(1)x+\frac{3}{2!}x^2+\frac{2}{3!}x^3+\ldots$$

$$=x+\frac{3}{2}x^2+\frac{1}{3}x^3+\ldots$$

18 a $\ln(1+\sin x) = \ln\left\{1+\left(x-\frac{x^3}{3!}+\ldots\right)\right\}$

$$= \left(x-\frac{x^3}{3!}+\ldots\right)-\frac{1}{2}\left(x-\frac{x^3}{3!}+\ldots\right)^2+\frac{1}{3}\left(x-\frac{x^3}{3!}+\ldots\right)^3-\frac{1}{4}\left(x-\frac{x^3}{3!}+\ldots\right)^4+\ldots$$

$$= \left(x-\frac{x^3}{6}+\ldots\right)-\frac{1}{2}\left(x^2-\frac{x^4}{3}+\ldots\right)+\frac{1}{3}\left(x^3+\ldots\right)-\frac{1}{4}(x^4+\ldots) \text{ no other terms necessary}$$

$$= x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}+\ldots$$

b $\int_0^{\frac{\pi}{6}} \ln(1+\sin x)\, dx \approx \int_0^{\frac{\pi}{6}}\left(x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}\right)dx$

$$\approx \left[\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{60}\right]_0^{\frac{\pi}{6}} = \frac{\pi^2}{72}-\frac{\pi^3}{1296}+\frac{\pi^4}{31104}-\frac{\pi^5}{466560} = 0.116 \ \ (3 \text{ d.p.})$$

19 a $f(x) = e^{\tan x} = e^{x+\frac{x^3}{3}+\ldots} = e^x \times e^{\frac{x^3}{3}}$ **(**As only terms up to x^3 are required, only first two terms of tan x are needed.**)**

$$= \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)\left(1+\frac{x^3}{3}+\ldots\right) \text{ no other terms required.}$$

$$= \left(1+\frac{x^3}{3}+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)$$

$$= 1+x+\frac{x^2}{2}+\frac{x^3}{2}+\ldots$$

b $e^{-\tan x} = e^{\tan(-x)}$, so replacing x by $-x$ in **a** gives

$$e^{-\tan x} = 1-x+\frac{x^2}{2}-\frac{x^3}{2}+\ldots$$

c Using the series expansions:

$$e^{\tan x} = 1+x+\frac{1}{2}x^2+\frac{1}{2}x^3+\ldots$$

$$e^x = 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots$$

$$\Rightarrow e^{\tan x}-e^x = \frac{1}{3}x^3+\ldots$$

$$\sin x - x = -\frac{1}{3!}x^3+\ldots$$

We can evaluate the limit:

$$\lim_{x\to 0}\frac{e^{\tan x}-e^x}{\sin x - x} = \lim_{x\to 0}\frac{\frac{1}{3}x^3+\ldots}{-\frac{1}{3!}x^3+\ldots}$$

$$= \lim_{x\to 0}\frac{\frac{1}{3}+\ldots}{-\frac{1}{6}+\ldots} = -2$$

20 We use the following series expansions:

$$x-\sin x=\frac{1}{3!}x^3-\frac{1}{5!}x^5\ldots$$

$$\Rightarrow x^2(x-\sin x)^2=\frac{1}{(3!)^2}x^8+\ldots=\frac{1}{36}x^8+\ldots$$

$$\cos x^2=1-\frac{1}{2}x^4+\frac{1}{4!}x^8+\ldots$$

$$\Rightarrow 2\cos x^2-2+x^4=\frac{2}{4!}x^8+\ldots=\frac{1}{12}x^8+\ldots$$

Then we can evaluate the limit:

$$\lim_{x\to 0}\frac{x^2(x-\sin x)^2}{2\cos x^2-2+x^4}=\lim_{x\to 0}\frac{\frac{1}{36}x^8+\ldots}{\frac{1}{12}x^8+\ldots}$$

$$=\lim_{x\to 0}\frac{\frac{1}{36}+\ldots}{\frac{1}{12}+\ldots}=\frac{1}{3}$$

21 a Differentiating the given differential equation with respect to x gives:

$$y\frac{d^3y}{dx^3}+\frac{dy}{dx}\frac{d^2y}{dx^2}+2\frac{dy}{dx}\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

$$\text{So }\frac{d^3y}{dx^3}=-\frac{1}{y}\left\{\frac{dy}{dx}\left(3\frac{d^2y}{dx^2}+1\right)\right\}$$

b Given that $y_0=1,\left(\frac{dy}{dx}\right)_0=1$ at $x=0$,

$$\left(\frac{d^2y}{dx^2}\right)_0+(1)^2+(1)=0,\text{ so }\left(\frac{d^2y}{dx^2}\right)_0=-2,$$

$$\text{And }\left(\frac{d^3y}{dx^3}\right)_0=-\frac{1}{(1)}(1)(3(-2)+1),\text{ so }\left(\frac{d^3y}{dx^3}\right)_0=5$$

$$\text{So }y=1+(1)x+\frac{(-2)}{2!}x^2+\frac{5}{3!}x^3+\ldots=1+x-x^2+\frac{5x^3}{6}+\ldots$$

c The approximation is best for small values of x (closed to 0): $x = 0.2$, therefore, would be acceptable, but not $x = 50$

22 a

$f(x)=\ln\cos x$	$f(0)=0$
$f'(x)=\frac{-\sin x}{\cos x}=-\tan x$	$f'(0)=0$
$f''(x)=-\sec^2 x$	$f''(0)=-1$
$f'''(x)=-2\sec^2 x\tan x$	$f'''(0)=0$
$f''''(x)=-2\sec^4 x-4\sec^2 x\tan^2 x$	$f''''(0)=-2$

Substituting into Maclaurin:

$$\ln\cos x=(-1)\frac{x^2}{2!}+(-2)\frac{x^4}{4!}+\ldots=-\frac{x^2}{2}-\frac{x^4}{12}-\ldots$$

22 b Using $1+\cos x = 2\cos^2\left(\frac{x}{2}\right)$, $\ln(1+\cos x) = \ln 2\cos^2\left(\frac{x}{2}\right) = \ln 2 + 2\ln\cos\left(\frac{x}{2}\right)$

so $\ln(1+\cos x) = \ln 2 + 2\left\{-\frac{1}{2}\left(\frac{x}{2}\right)^2 - \frac{1}{12}\left(\frac{x}{2}\right)^4 - \ldots\right\} = \ln 2 - \frac{x^2}{4} - \frac{x^4}{96} - \ldots$

c Using the series expansions derived above, we deduce that:

$\ln(2\cos x) = \ln 2 + \ln\cos x = \ln 2 - \frac{1}{2}x^2 + \ldots$

$\Rightarrow \ln(1+\cos x) - \ln(2\cos x) = \left(\ln 2 - \frac{1}{4}x^2 + \ldots\right)$

$- (\ln 2 - \frac{1}{2}x^2 + \ldots) = \frac{1}{4}x^2 + \ldots$

$1 - \cos x = \frac{1}{2}x^2 + \ldots$

Then we can calculate the limit:

$$\lim_{x\to 0}\frac{\ln(1+\cos x) - \ln(2\cos x)}{1-\cos x} = \lim_{x\to 0}\frac{\frac{1}{4}x^2 + \ldots}{\frac{1}{2}x^2 + \ldots}$$

$$= \lim_{x\to 0}\frac{\frac{1}{4} + \ldots}{\frac{1}{2} + \ldots} = \frac{1}{2}$$

23 a Let $y = 3^x$, then $\ln y = \ln 3^x = x\ln 3 \Rightarrow y = e^{x\ln 3}$ so $3^x = e^{x\ln 3}$

$$3^x = e^{x\ln 3} = 1 + (x\ln 3) + \frac{(x\ln 3)^2}{2!} + \frac{(x\ln 3)^3}{3!} + \ldots$$

$$= 1 + x\ln 3 + \frac{x^2(\ln 3)^2}{2} + \frac{x^3(\ln 3)^3}{6} + \ldots$$

b Put $x = \frac{1}{2}$: $\sqrt{3} \approx 1 + \frac{\ln 3}{2} + \frac{(\ln 3)^2}{8} + \frac{(\ln 3)^3}{48} = 1.73$ (3 s.f.)

24 a $f(x) = \operatorname{cosec} x$

$f'(x) = -\operatorname{cosec} x \cot x$

i $f''(x) = -\operatorname{cosec} x(-\operatorname{cosec}^2 x) + \cot x(\operatorname{cosec} x \cot x)$

$= \operatorname{cosec} x(\operatorname{cosec}^2 x + \cot^2 x)$

$= \operatorname{cosec} x\{\operatorname{cosec}^2 x + (\operatorname{cosec}^2 x - 1)\}$

$= \operatorname{cosec} x\{2\operatorname{cosec}^2 x - 1\}$

ii $f'''(x) = \operatorname{cosec} x(-4\operatorname{cosec}^2 x\cot x) - \operatorname{cosec} x\cot x(2\operatorname{cosec}^2 x - 1)$

$= -\operatorname{cosec} x\cot x(6\operatorname{cosec}^2 x - 1)$

24 b $\mathrm{f}\left(\frac{\pi}{4}\right)=\sqrt{2},\ \mathrm{f}'\left(\frac{\pi}{4}\right)=-\sqrt{2},\ \mathrm{f}''\left(\frac{\pi}{4}\right)=3\sqrt{2},\ \mathrm{f}'''\left(\frac{\pi}{4}\right)=-11\sqrt{2}$

Substituting all values into $y=y_0+(x-x_0)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x_0}+\frac{(x-x_0)^2}{2!}\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_{x_0}+\ldots$ with $x_0=\frac{\pi}{4}$

$$\operatorname{cosec} x=\sqrt{2}+(-\sqrt{2})\left(x-\frac{\pi}{4}\right)+\frac{(3\sqrt{2})}{2!}\left(x-\frac{\pi}{4}\right)^2+\frac{(-11\sqrt{2})}{3!}\left(x-\frac{\pi}{4}\right)^3+\ldots$$
$$=\sqrt{2}-\sqrt{2}\left(x-\frac{\pi}{4}\right)+\frac{3\sqrt{2}}{2}\left(x-\frac{\pi}{4}\right)^2-\frac{11\sqrt{2}}{6}\left(x-\frac{\pi}{4}\right)^3+\ldots$$

25 a We take $\mathrm{f}(x)=\ln\left(1+2\cos\left(\frac{\pi x}{2}\right)\right)$, then differentiating:

$$\mathrm{f}'(x)=\frac{1}{1+2\cos\left(\frac{\pi x}{2}\right)}\cdot\left(-2\cdot\tfrac{\pi}{2}\sin\left(\frac{\pi x}{2}\right)\right)=-\frac{\pi\sin\left(\frac{\pi x}{2}\right)}{1+2\cos\left(\frac{\pi x}{2}\right)}$$

$$\mathrm{f}''(x)=-\pi\left(\frac{\pi}{2}\frac{\cos\left(\frac{\pi x}{2}\right)}{\left(1+2\cos\left(\frac{\pi x}{2}\right)\right)}+\frac{\pi\sin^2\left(\frac{\pi x}{2}\right)}{\left(1+2\cos\left(\frac{\pi x}{2}\right)\right)^2}\right)$$
$$=-\frac{\pi^2\left(2+\cos\left(\frac{\pi x}{2}\right)\right)}{2\left(1+2\cos\left(\frac{\pi x}{2}\right)\right)^2}$$

b Evaluating the above at $x=1$, we find:
$\mathrm{f}(1)=\ln 1=0,\ \mathrm{f}'(1)=-\pi,\ \mathrm{f}''(1)=-\pi^2$

Hence the Taylor expansion about $x=1$ is:
$$\mathrm{f}(x)=-\pi(x-1)+\frac{1}{2!}(-\pi^2)(x-1)^2+\ldots$$
$$\Rightarrow \ln\left(1+2\cos\left(\frac{\pi x}{2}\right)\right)=-\pi(x-1)-\frac{\pi^2}{2}(x-1)^2+\ldots$$

c We use the Taylor expansion of $\ln(2-x)$ about $x=1$:
$$\ln(2-x)=-(x-1)-\frac{1}{2}(x-1)^2+\ldots$$

Then we can evaluate the limit:
$$\lim_{x\to1}\frac{\ln\left(1+2\cos\left(\frac{\pi x}{2}\right)\right)}{3\ln(2-x)}=\lim_{x\to1}\frac{-\pi(x-1)-\frac{\pi^2}{2}(x-1)^2+\ldots}{3(-(x-1)-\frac{1}{2}(x-1)^2+\ldots)}$$
$$=\lim_{x\to1}\frac{-\pi-\frac{\pi^2}{2}(x-1)+\ldots}{-3+\frac{1}{2}(x-1)+\ldots}=\frac{\pi}{3}$$

Challenge

a We have $\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x} = (-1)^{1+1}\frac{(1-1)!}{x^1}$, so holds for $n = 1$

Assume true for $n = k$ where $k \geqslant 1$

Then:

$$\frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}}\ln x = \frac{\mathrm{d}}{\mathrm{d}x}(-1)^{k+1}\cdot(k-1)!\cdot x^{-k}$$

$$= -k(-1)^{k+1}\cdot(k-1)!\cdot x^{-k-1} = (-1)^{(k+1)+1}\frac{((k+1)-1)!}{x^{k+1}}$$

So true for $n = k+1$

The result then follows by induction.

b Hence the Taylor series about $x = a, a > 0$ is:

$$\ln x = \ln a + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n-1)!}{n!a^n}(x-a)^n$$

$$= \ln a + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{na^n}(x-a)^n$$

c In our case we have $a_n = \frac{(-1)^{n+1}}{na^n}(x-a)^n$, so:

$$\frac{a_{n+1}}{a_n} = -(x-a)\cdot\frac{n}{a(n+1)} = -\frac{x-a}{a}\cdot\frac{1}{1+\frac{1}{n}}$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x-a}{a}\right| = \left|1-\tfrac{x}{a}\right|$$

where we have used $\lim_{n\to\infty}\frac{1}{n} = 0$

This is strictly less than 1 if and only if $0 < x < 2a$

So the ratio test shows that the Taylor series expansion converges for x such that $0 < x < 2a$

d We want to extend the range to include $x = 2a$

Setting $x = 2a$, we have an alternating series with $b_n = \frac{1}{n}$

Clearly, $b_n \geqslant 0$ for all $n \geqslant 1$, and $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n} = 0$

Finally, $b_n \geqslant b_{n+1} \Rightarrow \frac{n+1}{n} \geqslant 1 \Rightarrow 1+\frac{1}{n} \geqslant 1$

which is true for all $n \geqslant 1$

Hence, by the alternating series test, the Taylor series converges at $x = 2a$ Hence, the Taylor series converges for all $0 < x \leqslant 2a$ as required.