Taylor Series 6C

1 Differentiating $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x + 2y$, with respect to x, gives $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 1 + 2\frac{\mathrm{d}y}{\mathrm{d}x}$ **(1)**

Differentiating **(1)** gives $\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ **(2)**

Substituting $x_0 = 0, y_0 = 1$ into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x + 2y$, gives

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0 = 0 + 2\ (1), \text{ so } \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0 = 2$$

Substituting $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = \frac{1}{2}$ into **(1)** gives $\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)_0 = 1 + 2\left(\frac{1}{2}\right) = 2$

Substituting $\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0 = 2$ into **(2)** gives $\left(\frac{\mathrm{d}^4y}{\mathrm{d}x^4}\right)_0 = 2(2) = 4$

So using the Taylor expansion in the form where $x_0 = 0$, i.e. **ii**

$$y = 1 + \left(\frac{1}{2}\right)x + \frac{(2)}{2!}x^2 + \frac{(2)}{3!}x^3 + \frac{(4)}{4!}x^4 + \ldots = 1 + \frac{x}{2} + x^2 + \frac{x^3}{3} + \frac{x^4}{6} + \ldots$$

2 Differentiating $(1 + x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, gives

$$(1 + x^2)\frac{\mathrm{d}y^3}{\mathrm{d}x^3} + 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \textbf{(1)} \qquad \text{i.e. } (1 + x^2)\frac{\mathrm{d}y^3}{\mathrm{d}x^3} + 3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Substituting $x = 0$ and $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 1$ into $(1 + x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, gives $\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0 = 0$

Substituting $x = 0$, $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 1$ and $\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0 = 0$ into **(1)** gives $\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)_0 = -1$

So using the Taylor expansion in the form **ii**,

$$y = 0 + 1x + \frac{(0)}{2!}x^2 + \frac{(-1)}{3!}x^3 + \ldots = x - \frac{x^3}{6} + \ldots$$

3 Differentiating $\frac{dy}{dx}+y-e^x=0$, gives $\frac{d^2y}{dx^2}+\frac{dy}{dx}-e^x=0$ **(1)**

Differentiating **(1)** gives $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-e^x=0$ **(2)**

Substituting $x_0=0$ and $y_0=2$ into $\frac{dy}{dx}+y-e^x=0$, gives $\left(\frac{dy}{dx}\right)_0+2-1=0$, so $\left(\frac{dy}{dx}\right)_0=-1$

Substituting $x=0$, $\left(\frac{dy}{dx}\right)_0=-1$ into **(1)** gives $\left(\frac{d^2y}{dx^2}\right)_0+(-1)-(1)=0$ so $\left(\frac{d^2y}{dx^2}\right)_0=2$

Substituting $x=0$, $\left(\frac{d^2y}{dx^2}\right)_0=2$ into **(2)** gives $\left(\frac{d^3y}{dx^3}\right)_0+(2)-(1)=0$ so $\left(\frac{d^3y}{dx^3}\right)_0=-1$

Substituting into the Taylor series with $x_0=0$, gives

$$y=2+(-1)x+\frac{(2)}{2!}x^2+\frac{(-1)}{3!}x^3+\ldots$$
$$=2-x+x^2-\frac{x^3}{6}\ldots$$

4 Differentiating $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$ with respect to x gives

$\frac{d^3y}{dx^3}+x\frac{d^2y}{dx^2}+\frac{dy}{dx}+\frac{dy}{dx}=0$ **(1)**, i.e. $\frac{d^3y}{dx^3}+x\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$

Differentiating **(1)** gives

$\frac{d^4y}{dx^4}+x\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+2\frac{d^2y}{dx^2}=0$ **(2)**, i.e. $\frac{d^4y}{dx^4}+x\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}=0$

Substituting $x=0$, $y=1$ and $\frac{dy}{dx}=2$ into $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$ gives

$\left(\frac{d^2y}{dx^2}\right)_0+0(2)+1=0\Rightarrow\left(\frac{d^2y}{dx^2}\right)_0=-1$

Substituting $x=0$, $\left(\frac{dy}{dx}\right)_0=2$ and $\left(\frac{d^2y}{dx^2}\right)_0=-1$ into **(1)** gives

$\left(\frac{d^3y}{dx^3}\right)_0+0(-1)+2(2)=0$, so $\left(\frac{d^3y}{dx^3}\right)_0=-4$

Substituting $x=0$, $\left(\frac{dy}{dx}\right)_0=2$, $\left(\frac{d^2y}{dx^2}\right)_0=-1$ and $\left(\frac{d^3y}{dx^3}\right)_0=-4$ into **(2)** gives

$\left(\frac{d^4y}{dx^4}\right)_0+0(-4)+3(-1)=0$, so $\left(\frac{d^4y}{dx^4}\right)_0=3$

Substituting into the Taylor series with form **ii**, gives

$$y=1+2x+\frac{(-1)}{2!}x^2+\frac{(-4)}{3!}x^3+\frac{(3)}{4!}x^4+\ldots$$
$$=1+2x-\frac{1}{2}x^2-\frac{2}{3}x^3+\frac{1}{8}x^4+\ldots$$

5 Differentiating $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=3xy$ gives $\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}=3x\frac{dy}{dx}+3y$ **(1)**

Substituting $x_0=1,\ y_0=1$ and $\left(\frac{dy}{dx}\right)_1=-1$ into $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=3xy$ gives $\left(\frac{d^2y}{dx^2}\right)_1=5$

Substituting $x_0=1,\ y_0=1$ $\left(\frac{dy}{dx}\right)_1=-1$ and $\left(\frac{d^2y}{dx^2}\right)_1=5$ into **(1)** gives $\left(\frac{d^3y}{dx^3}\right)_1=-10$

Substituting into the form of the Taylor series form **i**, with $x_0=1$, gives

$$y=1+(-1)(x-1)+\frac{(5)}{2!}(x-1)^2+\frac{(-10)}{3!}(x-1)^3+\dots$$
$$=1-(x-1)+\frac{5}{2}(x-1)^2-\frac{5}{3}(x-1)^3+\dots$$

6 Differentiating $\frac{d^2y}{dx^2}+2y\frac{dy}{dx}+y^3=1+x$, twice with respect to x, gives

$$\frac{d^3y}{dx^3}+2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2+3y^2\frac{dy}{dx}=1 \quad \textbf{(1)}$$
$$\frac{d^4y}{dx^4}+2y\frac{d^3y}{dx^3}+2\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)+4\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)+3y^2\frac{d^2y}{dx^2}+6y\left(\frac{dy}{dx}\right)^2=0 \quad \textbf{(2)}$$

Substituting $x=0,\ y=1$ and $\frac{dy}{dx}=1$ into $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y^3=1+x$ gives $\left(\frac{d^2y}{dx^2}\right)_0=-2$

Substituting $y=1, \left(\frac{dy}{dx}\right)_0=1$ and $\left(\frac{d^2y}{dx^2}\right)_0=-2$ into **(1)** gives $\left(\frac{d^3y}{dx^3}\right)_0=0$

Substituting $y=1, \left(\frac{dy}{dx}\right)_0=1, \left(\frac{d^2y}{dx^2}\right)_0=-2, \left(\frac{d^3y}{dx^3}\right)_0=0$ into **(2)** gives $\left(\frac{d^4y}{dx^4}\right)_0=12$

So, using the Taylor series form **ii**, $y=1+1x+\frac{(-2)}{2!}x^2+\frac{(0)}{3!}x^3+\frac{(12)}{4!}x^4+\dots$

so $y=1+x-x^2+\frac{1}{2}x^4+\dots$

7 **a** Differentiating $(1+2x)\frac{dy}{dx}=x+2y^2$ with respect to x

$$\left\{(1+2x)\frac{d^2y}{dx^2}+2\frac{dy}{dx}\right\}=1+4y\frac{dy}{dx} \quad \textbf{(1)}$$

Differentiating **(1)** gives

$$\left\{(1+2x)\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}\right\}+\left\{2\frac{d^2y}{dx^2}\right\}=\left\{4y\frac{d^2y}{dx^2}+4\left(\frac{dy}{dx}\right)^2\right\}$$

$$\Rightarrow (1+2x)\frac{d^3y}{dx^3}+4(1-y)\frac{d^2y}{dx^2}=4\left(\frac{dy}{dx}\right)^2 \quad \textbf{(2)}$$

7 b Substituting $x_0 = 0$ and $y_0 = 1$ into $(1+2x)\frac{dy}{dx} = x + 2y^2$ gives $\left(\frac{dy}{dx}\right)_0 = 2(1) = 2$

Substituting known values into **(1)** gives

$$\left(\frac{d^2y}{dx^2}\right)_0 + 2(2) = 1 + 4(1)(2) \Rightarrow \left(\frac{d^2y}{dx^2}\right)_0 = 5$$

Substituting known values into **(2)** gives $\left(\frac{d^3y}{dx^3}\right)_0 = 4(2)^2 = 16$

So using $y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \ldots$

$$y = 1 + 2x + \frac{5}{2!}x^2 + \frac{16}{3!}x^3 + \ldots = 1 + 2x + \frac{5}{2}x^2 + \frac{8}{3}x^3 + \ldots$$

8 Differentiating $\sin x\frac{dy}{dx} + y\cos x = y^2$ with respect to *x*, gives

$$\left(\sin x\frac{d^2y}{dx^2} + \cos x\frac{dy}{dx}\right) + \left(-y\sin x + \cos x\frac{dy}{dx}\right) = 2y\frac{dy}{dx} \qquad \textbf{(1)}$$

or $\sin x\frac{d^2y}{dx^2} + 2\cos x\frac{dy}{dx} - y\sin x = 2y\frac{dy}{dx}$

Substituting $x_0 = \frac{\pi}{4}$, $y = \sqrt{2}$ into $\sin x\frac{dy}{dx} + y\cos x = y^2$ gives $\frac{1}{\sqrt{2}}\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} + \sqrt{2} \times \frac{1}{\sqrt{2}} = 2$

so $\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} = \sqrt{2}$

Substituting $x_0 = \frac{\pi}{4}$, $y_0\sqrt{2}$, $\left(\frac{dy}{dx}\right)_{\frac{\pi}{4}} = \sqrt{2}$ into **(1)** gives

$$\left\{\frac{1}{\sqrt{2}}\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} + 2\left(\frac{1}{\sqrt{2}}\right)(\sqrt{2}) - (\sqrt{2})\left(\frac{1}{\sqrt{2}}\right) = 2(\sqrt{2})(\sqrt{2})\right\}$$

So $\left\{\frac{1}{\sqrt{2}}\left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} + 2 - 1 = 4\right\} \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\frac{\pi}{4}} = 3\sqrt{2}$

Substituting all values into $y = y_0 + (x - x_0)\left(\frac{dy}{dx}\right)_{x_0} + \frac{(x-x_0)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_{x_0} + \ldots$

gives the series solution $y = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)^2 + \ldots$

9 a i Differentiating $\frac{\mathrm{d}y}{\mathrm{d}x}-x^2-y^2=0$ with respect to x, gives $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2y\frac{\mathrm{d}y}{\mathrm{d}x}-2x=0$ **(1)**

ii Differentiating **(1)** gives $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-2=0$

So $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=2$ **(2)**

b Differentiating **(2)** gives $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}-2y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)-4\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=0$

so $\frac{\mathrm{d}^4y}{\mathrm{d}x^4}-2y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-6\frac{\mathrm{d}y}{\mathrm{d}x}\times\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ **(3)**

c Substituting $x_0=0,\ y_0=1$, into $\frac{\mathrm{d}y}{\mathrm{d}x}-x^2-y^2=0$ gives

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0-0-1=1$, so $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0=1$

Substituting $x_0=0,\ y_0=1,\ \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0=1$ into **(1)** gives

$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0-2(1)(1)-2(0)=0$, so $\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0=2$

Substituting $y_0=1,\ \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0=1,\ \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0=2$ into **(2)** gives

$\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)_0-2(1)(2)-2(1)^2=2$, so $\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)_0=8$

Substituting $y_0=1,\ \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0=1,\ \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0=2$ and $\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)_0=8$ into **(3)** gives

$\left(\frac{\mathrm{d}^4y}{\mathrm{d}x^4}\right)_0-2(1)(8)-6(1)(2)=0$, so $\left(\frac{\mathrm{d}^4y}{\mathrm{d}x^4}\right)_0=28$

Substituting these values into the form of Taylor's series form **ii**, gives

$$y=1+(1)x+\frac{(2)}{2!}x^2+\frac{(8)}{3!}x^3+\frac{(28)}{4!}x^4+\ldots=1+x+x^2+\frac{4}{3}x^3+\frac{7}{6}x^4+\ldots$$

10 Differentiating $\cos x\frac{\mathrm{d}y}{\mathrm{d}x}+y\sin x+2y^3=0,$ **(1)** with respect to x, gives

$$\cos x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\cancel{\sin x\frac{\mathrm{d}y}{\mathrm{d}x}}+y\cos x+\cancel{\sin x\frac{\mathrm{d}y}{\mathrm{d}x}}+6y^2\frac{\mathrm{d}y}{\mathrm{d}x}=0, \qquad \textbf{(2)}$$

Differentiating again

$$\cos x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-\sin x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-y\sin x+\cos x\frac{\mathrm{d}y}{\mathrm{d}x}+6y^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+12y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0, \qquad \textbf{(3)}$$

Substituting $x_0=0,\ y_0=1$ into **(1)** gives $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0+2(1)=0$, so $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0=-2$

Substituting $x_0=0,\ y_0=1,\ \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0=-2$ into **(2)** gives

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0+1+6(1)(-2)=0,\text{ so }\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0=11$$

Substituting $x=0,\ y=1,\ \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0=-2,\ \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0=11$ into **(3)** gives

$$\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)_0+(1)(-2)+6(1)(11)+12(1)(-2)^2,\text{ so }\left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)_0=-112$$

Substituting these values into the form of Taylor's series form **ii**,

gives $y=1+(-2)x+\frac{11}{2!}x^2+\frac{(-112)}{3!}x^3+\ldots$

$$y=1-2x+\frac{11}{2}x^2-\frac{56}{3}x^3+\ldots$$

Ignoring terms in x^4 and higher powers, $y\approx 1-2x+\frac{11}{2}x^2-\frac{56}{3}x^3$

11 a We consider the differential equation:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=4x\frac{\mathrm{d}y}{\mathrm{d}x}-2y$$

Differentiating both sides:

$$\Rightarrow\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=4\frac{\mathrm{d}y}{\mathrm{d}x}+4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2\frac{\mathrm{d}y}{\mathrm{d}x}=2\frac{\mathrm{d}y}{\mathrm{d}x}+4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$
$$\Rightarrow\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=6\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4x\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$$
$$\Rightarrow\frac{\mathrm{d}^5y}{\mathrm{d}x^5}=6\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+4\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+4x\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=4x\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+10\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$$

i.e. $p=4,\ q=10$

11 b Now use the initial conditions given to find:

$$\frac{d^2y}{dx^2}(x=1) = 4 \cdot 1 \cdot 2 - 2 \cdot 2 = 4$$

$$\frac{d^3y}{dx^3}(x=1) = 2 \cdot 2 + 4 \cdot 1 \cdot 4 = 20$$

$$\frac{d^4y}{dx^4}(x=1) = 6 \cdot 4 + 4 \cdot 1 \cdot 20 = 104$$

$$\frac{d^5y}{dx^5}(x=1) = 10 \cdot 20 + 4 \cdot 1 \cdot 104 = 616$$

Plugging this into the Taylor expansion for $y(x)$, we see:

$$y(x) = 2 + 2(x-1) + \frac{1}{2!} \cdot 4(x-1)^2 + \frac{1}{3!} \cdot 20(x-1)^3$$

$$+ \frac{1}{4!} \cdot 104(x-1)^4 + \frac{1}{5!} \cdot 616(x-1)^5 + \ldots$$

$$\Rightarrow y(x) = 2 + 2(x-1) + 2(x-1)^2 + \frac{10}{3}(x-1)^3$$

$$+ \frac{13}{3}(x-1)^4 + \frac{77}{15}(x-1)^5 + \ldots$$