The *t*– formulae ME

1 a $t = \tan\frac{\theta}{2} = \frac{1}{2}$

$\cos\theta = \frac{1-t^2}{1+t^2} = \frac{1-\left(\frac{1}{2}\right)^2}{1+\left(\frac{1}{2}\right)^2} = \frac{\frac{3}{4}}{\frac{5}{4}} = \frac{3}{5}$

b $\sin\theta = \frac{2t}{1+t^2} = \frac{2\left(\frac{1}{2}\right)}{1+\left(\frac{1}{2}\right)^2} = \frac{1}{\frac{5}{4}} = \frac{4}{5}$

c $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$

So $\sec\theta + \tan\theta = \frac{1}{\cos\theta} + \tan\theta$

$= \frac{5}{3} + \frac{4}{3} = \frac{9}{3} = 3$

d $\sec\theta\operatorname{cosec}\theta = \left(\frac{1}{\cos\theta}\right)\left(\frac{1}{\sin\theta}\right)$

$= \left(\frac{5}{3}\right)\left(\frac{5}{4}\right) = \frac{25}{12}$

2 a $t = \tan\frac{\theta}{2} = \frac{4}{5}$

$\tan\theta = \frac{2t}{1-t^2} = \frac{2\left(\frac{4}{5}\right)}{1-\left(\frac{4}{5}\right)^2} = \frac{\frac{8}{5}}{\frac{9}{25}} = \frac{40}{9}$

b $\sec\theta = \frac{1}{\cos\theta} = \frac{1+t^2}{1-t^2} = \frac{1+\left(\frac{4}{5}\right)^2}{1-\left(\frac{4}{5}\right)^2} = \frac{\frac{41}{25}}{\frac{9}{25}} = \frac{41}{9}$

c $\sin\theta = \frac{2t}{1+t^2} = \frac{2\left(\frac{4}{5}\right)}{1+\left(\frac{4}{5}\right)^2} = \frac{\frac{8}{5}}{\frac{41}{25}} = \frac{40}{41}$

d $\cot\theta + \operatorname{cosec}\theta = \frac{1}{\tan\theta} + \frac{1}{\sin\theta}$

$= \frac{9}{40} + \frac{41}{40} = \frac{50}{40} = \frac{5}{4}$

3 a $t = \tan\theta = 3$

$\sin 2\theta = \frac{2t}{1+t^2} = \frac{2(3)}{1+3^2} = \frac{6}{10} = \frac{3}{5}$

b $\cos 2\theta = \frac{1-t^2}{1+t^2} = \frac{1-3^2}{1+3^2} = -\frac{8}{10} = -\frac{4}{5}$

c $\tan^2 2\theta = \left(\frac{\sin 2\theta}{\cos 2\theta}\right)^2 = \frac{\frac{3}{5}}{-\frac{4}{5}} = \left(-\frac{3}{4}\right)^2 = \frac{9}{16}$

d $\frac{\sec 2\theta}{\operatorname{cosec} 2\theta + \tan 2\theta} = \frac{-\frac{5}{4}}{\frac{5}{3}+\left(-\frac{4}{3}\right)} = -\frac{\frac{5}{4}}{\frac{1}{3}} = -\frac{15}{4}$

4 a $t = \tan\theta = -2$

$\tan 2\theta = \frac{2t}{1-t^2} = \frac{2(-2)}{1-(-2)^2} = \frac{-4}{-3} = \frac{4}{3}$

b $\cos\theta = \frac{1-t^2}{1+t^2} = \frac{1-(-2)^2}{1+(-2)^2} = -\frac{3}{5}$

and $\sin 2\theta = \frac{2t}{1+t^2} = \frac{2(-2)}{1+(-2)^2} = -\frac{4}{5}$

So $\sec 2\theta\operatorname{cosec} 2\theta = \left(\frac{1}{\cos 2\theta}\right)\left(\frac{1}{\sin 2\theta}\right)$

$= \left(-\frac{5}{4}\right)\left(-\frac{5}{3}\right) = \frac{25}{12}$

c $\sec^2 2\theta = \left(\frac{1}{\cos 2\theta}\right)^2 = \left(-\frac{5}{3}\right)^2 = \frac{25}{9}$

d $\cot 2\theta + \tan 2\theta = \frac{1}{\tan 2\theta} + \tan 2\theta$

$= \frac{3}{4} + \frac{4}{3} = \frac{9}{12} + \frac{16}{12} = \frac{25}{12}$

5 a Substitute $t = \tan\frac{\theta}{2}$

$\sec^2\theta - 1 = \frac{1}{\cos^2\theta} - 1 = \left(\frac{1+t^2}{1-t^2}\right) - 1$

$= \frac{\left(1+t^2\right)^2}{\left(1-t^2\right)^2} - \frac{\left(1-t^2\right)^2}{\left(1-t^2\right)^2}$

$= \frac{1+2t^2+t^4-\left(1-2t^2+t^4\right)}{\left(1-t^2\right)^2} = \frac{4t^2}{\left(1-t^2\right)^2}$

$= \left(\frac{2t}{1-t^2}\right)^2 = \tan^2\theta$

5 b $\sec\theta = -\dfrac{2\sqrt{2}}{1+\sqrt{3}}$

$\Rightarrow \sec^2\theta = \dfrac{8}{4+2\sqrt{3}} = \dfrac{4}{2+\sqrt{3}}$

So $\sec^2\theta - 1 = \dfrac{4}{2+\sqrt{3}} - 1$

$= \dfrac{4}{2+\sqrt{3}} - \dfrac{2+\sqrt{3}}{2+\sqrt{3}}$

$= \dfrac{2-\sqrt{3}}{2+\sqrt{3}} = \dfrac{4-2\sqrt{3}}{4+2\sqrt{3}}$

$= \left(\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\right)^2 = \tan^2\theta$

So $\tan\theta = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

Note that the positive root is taken, since $\pi \leqslant \theta \leqslant \dfrac{3\pi}{2}$ and $\tan\theta$ is positive in this range.

c $\sin 2\theta = \dfrac{2t}{1+t^2} = \dfrac{2\left(\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\right)}{1+\dfrac{4-2\sqrt{3}}{4+2\sqrt{3}}}$

$= \dfrac{\dfrac{2\sqrt{3}-2}{\sqrt{3}+1}}{\dfrac{8}{4+2\sqrt{3}}} = \left(\dfrac{2\sqrt{3}-2}{\sqrt{3}+1}\right)\left(\dfrac{4+2\sqrt{3}}{8}\right)$

$= \dfrac{4+4\sqrt{3}}{8+8\sqrt{3}} = \dfrac{4\left(1+\sqrt{3}\right)}{8\left(1+\sqrt{3}\right)} = \dfrac{4}{8} = \dfrac{1}{2}$

$\cos 2\theta = \dfrac{1-t^2}{1+t^2} = \dfrac{1-\left(\dfrac{4-2\sqrt{3}}{4+2\sqrt{3}}\right)}{1+\left(\dfrac{4-2\sqrt{3}}{4+2\sqrt{3}}\right)}$

$= \dfrac{\dfrac{4\sqrt{3}}{4+2\sqrt{3}}}{\dfrac{8}{4+2\sqrt{3}}} = \dfrac{4\sqrt{3}}{8} = \dfrac{\sqrt{3}}{2}$

d $\tan 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{1}{\sqrt{3}}$

So $2\theta = \dfrac{\pi}{6}$ or $\dfrac{7\pi}{6}$ or $\dfrac{13\pi}{6}\ldots$

So $\theta = \dfrac{\pi}{12}$ or $\dfrac{7\pi}{12}$ or $\dfrac{13\pi}{12}\ldots$

Since $\pi \leqslant \theta \leqslant \dfrac{3\pi}{2}$, you have that

$\theta = \dfrac{13\pi}{12}$

6 a $t = \tan\dfrac{\pi}{8}$

$\cos\dfrac{\pi}{4} = \dfrac{1-t^2}{1+t^2}$ and $\sin\dfrac{\pi}{4} = \dfrac{2t}{1+t^2}$

$\tan\dfrac{\pi}{4} = \dfrac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = \left(\dfrac{2t}{1+t^2}\right)\left(\dfrac{1+t^2}{1-t^2}\right) = \dfrac{2t}{1-t^2}$

Also, $\tan\dfrac{\pi}{4} = 1$, so

$\dfrac{2t}{1-t^2} = 1$

So $2t = 1 - t^2$

$t^2 + 2t - 1 = 0$

$t = \dfrac{-2 \pm \sqrt{8}}{2}$

$t = -1 \pm \sqrt{2}$

Since $t = \tan\dfrac{\pi}{8} > 0$, you have that

$t = -1 + \sqrt{2}$

6 b $1+\tan^2\frac{\pi}{8}=1+\left(\sqrt{2}-1\right)^2$

$=1+\left(3-2\sqrt{2}\right)=4-2\sqrt{2}$

So $\sec^2\frac{\pi}{8}=4-2\sqrt{2}$

and $\sec\frac{\pi}{8}=\sqrt{4-2\sqrt{2}}$

$\cos^2\frac{\pi}{8}=\frac{1}{\sec^2\frac{\pi}{8}}=\frac{1}{4-2\sqrt{2}}$

$=\left(\frac{1}{4-2\sqrt{2}}\right)\left(\frac{4+2\sqrt{2}}{4+2\sqrt{2}}\right)=\frac{4+2\sqrt{2}}{8}$

$=\frac{2+\sqrt{2}}{4}$

So $\cos\frac{\pi}{8}=\frac{\sqrt{2+\sqrt{2}}}{2}$

$\sin^2\frac{\pi}{8}=1-\cos^2\frac{\pi}{8}$

$=1-\frac{2+\sqrt{2}}{4}=\frac{2-\sqrt{2}}{4}$

So $\sin\frac{\pi}{8}=\frac{\sqrt{2-\sqrt{2}}}{2}$

7 $\frac{1+\sin x-\cos x}{\sin x+\cos x-1}=\frac{\frac{1+t^2}{1+t^2}+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}-\frac{1+t^2}{1+t^2}}$

$=\frac{1+t^2+2t-1+t^2}{2t+1-t^2-1-t^2}=\frac{2t^2+2t}{-2t^2+2t}$

$=\frac{2t(t+1)}{2t(1-t)}=\frac{1+t}{1-t}$

Also $\frac{1+\sin x}{\cos x}=\frac{1+\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}}=\frac{1+t^2+2t}{1-t^2}$

$=\frac{(1+t)^2}{(1+t)(1-t)}=\frac{1+t}{1-t}$

Hence $\frac{1+\sin x-\cos x}{\sin x+\cos x-1}=\frac{1+\sin x}{\cos x}$

8 $\tan^2\theta-\sin^2\theta=\left(\frac{2t}{1-t^2}\right)^2-\left(\frac{2t}{1+t^2}\right)^2$

$=\frac{4t^2\left(1+t^2\right)^2-4t^2\left(1+t^2\right)^2}{\left(1-t^2\right)^2\left(1+t^2\right)^2}$

$=\frac{4t^2\left[\left(1+t^2\right)^2-\left(1-t^2\right)^2\right]}{\left(1-t^2\right)^2\left(1+t^2\right)^2}$

$=\frac{4t^2\left[1+2t^2+t^4-\left(1-2t^2+t^4\right)\right]}{\left(1-t^2\right)^2\left(1+t^2\right)^2}$

$=\left(\frac{4t^2}{\left(1-t^2\right)^2}\right)\left(\frac{4t^2}{\left(1+t^2\right)^2}\right)$

$=\left(\frac{2t}{1-t^2}\right)^2\left(\frac{2t}{1+t^2}\right)^2$

$=\tan^2\theta\sin^2\theta$

9 $\sin\theta\cos\theta\tan\theta=\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)\left(\frac{2t}{1-t^2}\right)$

$=\frac{4t^2}{\left(1+t^2\right)^2}$

Also, $1-\cos^2\theta=1-\left(\frac{1-t^2}{1+t^2}\right)^2$

$=\frac{\left(1+t^2\right)^2-\left(1-t^2\right)^2}{\left(1+t^2\right)^2}$

$=\frac{1+2t^2+t^4-\left(1-2t^2+t^4\right)}{\left(1+t^2\right)^2}=\frac{4t^2}{\left(1+t^2\right)^2}$

Hence $\sin\theta\cos\theta\tan\theta=1-\cos^2\theta$

10 $$\frac{1+\sin\theta}{1-\sin\theta}-\frac{1-\sin\theta}{1+\sin\theta}=\frac{1+\frac{2t}{1+t^2}}{1-\frac{2t}{1+t^2}}-\frac{1-\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}}$$

$$=\left(\frac{1+2t+t^2}{1-2t+t^2}\right)-\left(\frac{1-2t+t^2}{1+2t+t^2}\right)$$

$$=\frac{(1+t)^2}{(1-t)^2}-\frac{(1-t)^2}{(1+t)^2}$$

$$=\frac{(1+t)^4-(1-t)^4}{(1-t)^2(1+t)^2}$$

$$=\frac{(1+t)^4-(1-t)^4}{(1+t)(1-t)(1+t)(1-t)}$$

$$=\frac{(1+t)^4-(1-t)^4}{(1-t^2)(1-t^2)}$$

$$=\frac{(1+t)^4-(1-t)^4}{(1-t^2)^2}$$

$$=\frac{(1+4t+6t^2+4t^3+t^4)-(1-4t+6t^2-4t^3+t^4)}{(1-t^2)^2}$$

$$=\frac{8t+8t^3}{(1-t^2)^2}=\frac{8t(1+t^2)}{(1-t^2)^2}$$

Also, $4\tan\theta\sec\theta=4\left(\frac{2t}{1-t^2}\right)\left(\frac{1+t^2}{1-t^2}\right)$

$$=\frac{8t(1+t^2)}{(1-t^2)^2}$$

Hence $\frac{1+\sin\theta}{1-\sin\theta}-\frac{1-\sin\theta}{1+\sin\theta}=4\tan\theta\sec\theta$

11 $$\frac{1+\tan^2 x}{1-\tan^2 x}=\frac{1+\left(\frac{2t}{1-t^2}\right)^2}{1-\left(\frac{2t}{1-t^2}\right)^2}$$

$$=\frac{(1-t^2)^2+4t^2}{(1-t^2)^2-4t^2}$$

$$=\frac{1-2t^2+t^4+4t^2}{(1-t^2)^2-4t^2}=\frac{1+2t^2+t^4}{(1-t^2)^2-4t^2}$$

$$=\frac{(1+t^2)^2}{(1-t^2)^2-4t^2}$$

Also,

$$\frac{1}{\cos^2 x-\sin^2 x}=\frac{1}{\left(\frac{1-t^2}{1+t^2}\right)^2-\left(\frac{2t}{1+t^2}\right)^2}$$

$$=\frac{1}{\frac{(1-t^2)^2-4t^2}{(1+t^2)^2}}$$

$$=\frac{(1+t^2)^2}{(1-t^2)^2-4t^2}$$

Hence $\frac{1+\tan^2 x}{1-\tan^2 x}=\frac{1}{\cos^2 x-\sin^2 x}$

12 $\frac{1}{1-\sin\theta}-\frac{1}{1+\sin\theta}$

$=\frac{1}{1-\frac{2t}{1+t^2}}-\frac{1}{1+\frac{2t}{1+t^2}}$

$=\left(\frac{1+t^2}{1-2t+t^2}\right)-\left(\frac{1+t^2}{1+2t+t^2}\right)$

$=\frac{1+t^2}{(1-t)^2}-\frac{1+t^2}{(1+t)^2}$

$=\frac{(1+t^2)(1+t)^2-(1+t^2)(1-t)^2}{(1-t)^2(1+t)^2}$

$=\frac{(1+t^2)\left((1+t)^2-(1-t)^2\right)}{(1+t)(1-t)(1+t)(1-t)}$

$=\frac{(1+t^2)\left((1+t)^2-(1-t)^2\right)}{(1-t^2)(1-t^2)}$

$=\frac{(1+t^2)\left((1+t)^2-(1-t)^2\right)}{(1-t^2)^2}$

$=\frac{(1+t^2)\left([(1+t)+(1-t)][(1+t)-(1-t)]\right)}{(1-t^2)^2}$

$=\frac{(1+t^2)(2)(2t)}{(1-t^2)^2}$

$=\frac{4t(1+t^2)}{(1-t^2)^2}$

$=2\left(\frac{2t}{1-t^2}\right)\left(\frac{1+t^2}{1-t^2}\right)$

$=2\tan\theta\sec\theta$

13 $\tan\theta+\frac{\cos\theta}{1+\sin\theta}$

$=\frac{2t}{1-t^2}+\frac{\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}}$

$=\frac{2t}{1-t^2}+\frac{1-t^2}{1+2t+t^2}$

$=\frac{2t}{1-t^2}+\frac{1-t^2}{(1+t)^2}$

$=\frac{2t}{(1+t)(1-t)}+\frac{1-t^2}{(1+t)^2}$

$=\frac{2t(1+t)+(1-t)(1-t^2)}{(1+t)^2(1-t)}$

$=\frac{2t+2t^2+1-t-t^2+t^3}{(1+t)^2(1-t)}$

$=\frac{1+t+t^2+t^3}{(1+t)^2(1-t)}$

$=\frac{(1+t^2)(1+t)}{(1+t)^2(1-t)}$

$=\frac{1+t^2}{(1+t)(1-t)}$

$=\frac{1+t^2}{1-t^2}$

$=\sec\theta$

14 $(\sin\theta+\cos\theta)(\tan\theta+\cot\theta)$

$=\left(\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}\right)\left(\frac{2t}{1-t^2}+\frac{1-t^2}{2t}\right)$

$=\left(\frac{2t+1-t^2}{1+t^2}\right)\left(\frac{4t^2+(1-t^2)^2}{2t(1-t^2)}\right)$

$=\left(\frac{2t+1-t^2}{1+t^2}\right)\left(\frac{(1+t^2)^2}{2t(1-t^2)}\right)$

$=\frac{(2t+1-t^2)(1+t^2)}{2t(1-t^2)}$

$=\frac{1+2t+2t^3-t^4}{2t(1-t^2)}$

Also, $\sec\theta+\operatorname{cosec}\theta$

$=\frac{1+t^2}{1-t^2}+\frac{1+t^2}{2t}$

$=\frac{2t(1+t^2)+(1+t^2)(1-t^2)}{2t(1-t^2)}$

$=\frac{2t+2t^3+1-t^4}{2t(1-t^2)}$

$=\frac{1+2t+2t^3-t^4}{2t(1-t^2)}$

Hence,

$(\sin\theta+\cos\theta)(\tan\theta+\cot\theta)$

$=\sec\theta+\operatorname{cosec}\theta$

15 a $3\cos x-\sin x=-1$

Substitute $t=\tan\frac{x}{2}$

$3\left(\frac{1-t^2}{1+t^2}\right)-\frac{2t}{1+t^2}=-1$

$3(1-t^2)-2t=-1-t^2$

$3-3t^2-2t=-1-t^2$

$2t^2+2t-4=0$

$t^2+t-2=0$

b $(t+2)(t-1)=0$

$t=-2$ or $t=1$

$\tan\frac{x}{2}=-2$

$\frac{x}{2}=\pi-1.107\ldots$

$x=4.07$

$\tan\frac{x}{2}=1$

$\frac{x}{2}=\frac{\pi}{4}$

$x=\frac{\pi}{2}=1.57$

16 a $\sin\theta+\cos\theta=-\frac{1}{5}$

Substitute $t=\tan\frac{\theta}{2}$

$\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}-=-\frac{1}{5}$

$10t+5(1-t^2)=-(1+t^2)$

$10t+5-5t^2=-1-t^2$

$4t^2-10t-6=0$

$2t^2-5t-3=0$

b $(2t+1)(t-3)=0$

$t=-\frac{1}{2}$ or $t=3$

$\tan\frac{\theta}{2}=-\frac{1}{2}$

$\frac{\theta}{2}=\pi-0.463\ldots$

$\theta=5.36$

$\tan\frac{\theta}{2}=3$

$\frac{\theta}{2}=1.24\ldots$

$\theta=2.50$

17 $6\tan\theta + 12\sin\theta + \cos\theta = 1$

Substitute $t = \tan\dfrac{\theta}{2}$

$6\left(\dfrac{2t}{1-t^2}\right) + 12\left(\dfrac{2t}{1+t^2}\right) + \dfrac{1-t^2}{1+t^2} = 1$

$\dfrac{12t}{1-t^2} + \dfrac{24t}{1+t^2} + \dfrac{1-t^2}{1+t^2} = 1$

$12t\left(1+t^2\right) + 24t\left(1-t^2\right) + \left(1-t^2\right)^2$
$= \left(1+t^2\right)\left(1-t^2\right)$

$12t + 12t^3 + 24t - 24t^3 + 1 - 2t^2 + t^4$
$= 1 - t^4$
$1 + 36t - 2t^2 - 12t^3 + t^4 = 1 - t^4$
$2t^4 - 12t^3 - 2t^2 + 36t = 0$
$t^4 - 6t^3 - t^2 + 18t = 0$
$t\left(t^3 - 6t^2 - t + 18\right) = 0$

Let $f(t) = t^3 - 6t^2 - t + 18$
$f(2) = 2^3 - 6\times 2^2 - 2 + 18 = 0$
So $t-2$ is a factor of $f(t)$

$f(t) = (t-2)\left(t^2 - kt - 9\right)$

Coefficients of t^2:

$-6 = -2 - k$
$k = 4$

$f(t) = (t-2)\left(t^2 - 4t - 9\right)$

So $t(t-2)\left(t^2 - 4t - 9\right) = 0$

If $t = 0$
$\tan\dfrac{\theta}{2} = 0$
$\dfrac{\theta}{2} = 0$ or $\dfrac{\theta}{2} = \pi$
$\theta = 0$ or $\theta = 2\pi = 6.28$

If $t - 2 = 0$
$\tan\dfrac{\theta}{2} = 2$
$\dfrac{\theta}{2} = 1.107\ldots$
$\theta = 2.21$

If $t^2 - 4t - 9 = 0$
$t = \dfrac{4 \pm \sqrt{52}}{2} = 2 \pm \sqrt{13}$

$\tan\dfrac{\theta}{2} = 2 + \sqrt{13}$
$\dfrac{\theta}{2} = 1.394\ldots$
$\theta = 2.79$

$\tan\dfrac{\theta}{2} = 2 - \sqrt{13}$
$\dfrac{\theta}{2} = \pi - 1.0137\ldots$
$\theta = 4.26$

The solutions are therefore
$\theta = 0,\ 2.21,\ 2.79,\ 4.26,\ 6.28$

18 a $5\cot x + 4\operatorname{cosec} x = \dfrac{9}{4}$

$5\left(\dfrac{1-t^2}{2t}\right) + 4\left(\dfrac{1+t^2}{2t}\right) = \dfrac{9}{4}$

$5\left(1-t^2\right) + t\left(1+t^2\right) = \dfrac{18t}{4}$

$5 - 5t^2 + 4 + 4t^2 = \dfrac{9t}{2}$

$10 - 10t^2 + 8 + 8t^2 = 9t$
$2t^2 + 9t - 18 = 0$

b $(2t-3)(t+6) = 0$

$t = \dfrac{3}{2}$ or $t = -6$

$\tan\dfrac{x}{2} = \dfrac{3}{2}$

$\dfrac{x}{2} = 0.982\ldots$
$x = 1.97$

$\tan\dfrac{x}{2} = -6$

$\dfrac{x}{2} = \pi - 1.40\ldots$
$x = 3.47$

19 a $p(x) = 8\sin 5x + 16\cos 5x - 4\sin 10x + \frac{16}{3}\cos 10x + 100$

$p'(x) = 40\cos 5x - 80\sin 5x - 40\cos 10x - \frac{160}{3}\sin 10x$

$p'(x)$

$= 10\left[4\cos 5x - 8\sin 5x - 4\cos 10x - \frac{16}{3}\sin 10x\right]$

$= 10\left[4\cos 5x - 8\sin 5x - 4\cos 10x - \frac{32}{3}\sin 5x\cos 5x\right]$

$= 10\left[4\cos 5x - 8\sin 5x - 4\left(\cos^2 5x - \sin^2 5x\right) - \frac{32}{3}\sin 5x\cos 5x\right]$

$= 10\left[4\cos 5x - 8\sin 5x - 4\cos^2 5x + 4\sin^2 5x - \frac{32}{3}\sin 5x\cos 5x\right]$

Substituting $t = \tan\frac{5x}{2}$

$p'(x)$

$= 10\left[4\left(\frac{1-t^2}{1+t^2}\right) - 8\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right)^2 + 4\left(\frac{2t}{1+t^2}\right)^2 - \frac{32}{3}\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)\right]$

$= \frac{10}{\left(1+t^2\right)^2}\left[4\left(1-t^2\right)\left(1+t^2\right) - 16t\left(1+t^2\right) - 4\left(1-t^2\right)^2 + 4(2t)^2 - \frac{64t\left(1-t^2\right)}{3}\right]$

$= \frac{10}{\left(1+t^2\right)^2}\left[4\left(1-t^4\right) - 16t - 16t^3 - 4\left(1-2t^2+t^4\right) + 16t^2 - \frac{64t}{3} + \frac{64t^3}{3}\right]$

$= \frac{10}{\left(1+t^2\right)^2}\left[4 - 4t^4 - 16t - 16t^3 - 4 + 8t^2 - 4t^4 + 16t^2 - \frac{64t}{3} + \frac{64t^3}{3}\right]$

Hence

$p'(x) = \frac{10}{\left(1+t^2\right)^2}\left[-\frac{112t}{3} + 24t^2 + \frac{16t^3}{3} - 8t^4\right]$

$= \frac{10}{\left(1+t^2\right)^2}\left[-\frac{112t}{3} + \frac{72t^2}{3} + \frac{16t^3}{3} - \frac{24t^4}{3}\right]$

$= \frac{10}{3\left(1+t^2\right)^2}\left[-112t + 72t^2 + 16t^3 - 24t^4\right]$

$= -\frac{80}{3\left(1+t^2\right)^2}\left[14t - 9t^2 - 2t^3 + 3t^4\right]$

So $p'(x) = -\frac{80t}{3\left(1+t^2\right)^2}\left[3t^3 - 2t^2 - 9t + 14\right]$

Let $f(t) = 3t^3 - 2t^2 - 9t + 14$

Then $f(-2) = 3(-2)^3 - 2(-2)^2 - 9(-12) + 14$
$= -24 - 8 + 18 + 14$
$= 0$

Therefore $(t+2)$ is a factor of $f(t)$

Writing $3t^3 - 2t^2 - 9t + 14 = (t+2)\left(3t^2 + kt + 7\right)$

Equating coefficients of t^2 :
$-2 = k + 6 \Rightarrow k = 8$
So $f(t) = (t+2)\left(3t^2 - 8t + 7\right)$

Therefore $p'(x) = \frac{-80t(t+2)\left(3t^2 - 8t + 7\right)}{3\left(1+t^2\right)^2}$

b The maxima and minima do not change, whereas we might expect blood pressure to vary with each heartbeat.

Also this model has a fixed period, whereas heart rates are not constant, and will vary, with, for example, physical activity. This model doesn't capture changing heart rates.

c At a low pressure point, we have $\frac{\mathrm{d}p}{\mathrm{d}x} = 0$.

From part a), this happens when $t = 0$ or $t = -2$

We can see from the figure that the solution $t = 0$ corresponds to the maximum at $x = 0$

Therefore $t = \tan\frac{5x}{2} = -2$

$\frac{5x}{2} = \pi - 1.107\ldots$

$x = \frac{2}{5}(\pi - 1.107\ldots)$

So $x = 0.814$ (3 d.p.)

Challenge

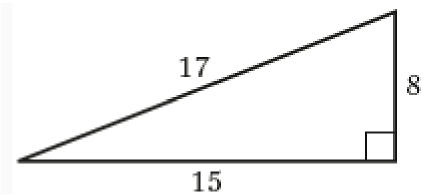
a $\tan\frac{\theta}{2} = \frac{1}{4}$

So $\tan\theta = \frac{2t}{1-t^2} = \frac{2\left(\frac{1}{4}\right)}{1-\left(\frac{1}{4}\right)^2} = \frac{\frac{1}{2}}{\frac{15}{16}} = \frac{8}{15}$

$\sin\theta = \frac{2t}{1+t^2} = \frac{2\left(\frac{1}{4}\right)}{1+\left(\frac{1}{4}\right)^2} = \frac{\frac{1}{2}}{\frac{17}{16}} = \frac{8}{17}$

$\cos\theta = \frac{1-t^2}{1+t^2} = \frac{1-\left(\frac{1}{4}\right)^2}{1+\left(\frac{1}{4}\right)^2} = \frac{\frac{15}{16}}{\frac{17}{16}} = \frac{15}{17}$

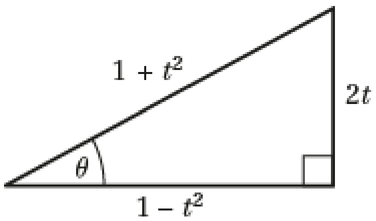
b



c Given $t = \tan\frac{\theta}{2}$ is rational, we can write

$t = \frac{n}{m}$, where n and m are integers and have no common factors (apart from 1).

So we can construct the following triangle:



Multiplying each side by m^2 will give a resulting triangle having sides $m^2 - n^2$, $m^2 + n^2$ and $2mn$. (i.e. each side will be of integer length)

This new triangle will also be similar to the above triangle, therefore the angle θ will remain the same.

Since m and n have no common factors, neither will the sides in this new triangle.

d Using the above construction, every rational value of $\tan\frac{\theta}{2}$ between 0 and 1 gives rise to a primitive Pythagorean triple.

Note that the same triple is generated by triangles with acute angles θ and $90-\theta$, so we obtain a unique triple for every value of θ such that $0 \leqslant \theta \leqslant \frac{\pi}{4}$

However, there are infinitely many values of θ in this range such that $t = \tan\frac{\theta}{2}$ is rational.

Therefore there are infinitely many primitive Pythagorean triples.