The *t*– formulae 5D

1 a $s = 10 - 5\sin x - 12\cos x$

$\frac{\mathrm{ds}}{\mathrm{dx}} = -5\cos x + 12\sin x$

Substitute $t = \tan\frac{x}{2}$

$\frac{\mathrm{ds}}{\mathrm{dx}} = -5\left(\frac{1-t^2}{1+t^2}\right) + 12\left(\frac{2t}{1+t^2}\right)$

$= \frac{-5+5t^2+24t}{1+t^2}$

$= \frac{1}{1+t^2}\left(5t^2+24t-5\right)$

b The displacement is minimal when $\frac{\mathrm{d}s}{\mathrm{d}x} = 0$:

$5t^2 + 24t - 5 = 0$

$(5t-1)(t+5)$

$t = \frac{1}{5}$ or $t = -5$

$\frac{\mathrm{d}^2 s}{\mathrm{d}x^2} = 5\sin x + 12\cos x$

$= 5\left(\frac{2t}{1+t^2}\right) + 12\left(\frac{1-t^2}{1+t^2}\right)$

$= \frac{10t+12-12t^2}{1+t^2}$

At $t = \frac{1}{5}$, $\frac{\mathrm{d}^2\mathrm{s}}{\mathrm{dx}^2} > 0$,
so the displacement is minimised at this point.

$\tan\frac{x}{2} = \frac{1}{5}$

$\frac{x}{2} = \arctan\frac{1}{5} + n\pi$

$x = 2\arctan\frac{1}{5} + 2n\pi$

So in the range $0 \leqslant x \leqslant 2\pi$,

$x = 2\arctan\frac{1}{5} = 0.395$ (3 s.f.)

2 a $s = 1 + 2\sin x - \cos 2x$

$\frac{\mathrm{d}s}{\mathrm{d}x} = 2\cos x + 2\sin 2x$

$= 2\cos x + 4\sin x\cos x$

Substitute $t = \tan\frac{x}{2}$

$\frac{\mathrm{d}s}{\mathrm{d}x} = 2\left(\frac{1-t^2}{1+t^2}\right) + 4\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)$

$= \frac{2(1-t^2)(1+t^2)}{(1+t^2)^2} + \frac{8t(1-t^2)}{(1+t^2)^2}$

$= \frac{2(1-t^2)(1+t^2)+8t(1-t^2)}{(1+t^2)^2}$

$= \frac{2(1-t^2)(1+4t+t^2)}{(1+t^2)^2}$

2 b Particle is stationary when $\frac{\mathrm{d}s}{\mathrm{d}x} = 0$

$1 - t^2 = 0$

$t = 1$ or $t = -1$

$\tan\frac{\theta}{2} = 1$

$\frac{\theta}{2} = n\pi + \frac{\pi}{4}$

$\theta = 2n\pi + \frac{\pi}{2}$

So in the given range, $\theta = \frac{\pi}{2}$

$\tan\frac{\theta}{2} = -1$

$\frac{\theta}{2} = n\pi - \frac{\pi}{4}$

$\theta = 2n\pi - \frac{\pi}{2}$

So in the given range, $\theta = \frac{3\pi}{2}$

$t^2 + 4t + 1 = 0$

$t = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$

$\tan\frac{\theta}{2} = -2 + \sqrt{3}$

$\frac{\theta}{2} = \pi - 0.267\ldots$

$\theta = \frac{11\pi}{6} = 5.76$ (2 d.p.)

$\tan\frac{\theta}{2} = -2 - \sqrt{3}$

$\frac{\theta}{2} = \pi - 1.308\ldots$

$\theta = \frac{7\pi}{6} = 3.67$ (2 d.p.)

3 a $h(x) = 3\sin 2x - 4\cos 2x + 25$

$\frac{\mathrm{d}h}{\mathrm{d}x} = 6\cos 2x + 8\sin 2x$

Substitute $t = \tan x$

$\frac{\mathrm{d}h}{\mathrm{d}x} = 6\left(\frac{1-t^2}{1+t^2}\right) + 8\left(\frac{2t}{1+t^2}\right)$

$= \frac{6 - 6t^2 + 16t}{1+t^2}$

$= \frac{-2}{1+t^2}\left(3t^2 - 8t - 3\right)$

b $\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{-2}{1+t^2}\left(3t^2 - 8t - 3\right)$

$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{-2}{1+t^2}(3t+1)(t-3)$

$\frac{\mathrm{d}h}{\mathrm{d}x} = 0 \Rightarrow (3t+1)(t-3) = 0$

$t = -\frac{1}{3}$ or $t = 3$

Taking $t = 3$, this gives either consecutive minimum or consecutive maximum points.

$\tan x = 3$

$x = n\pi + \arctan 3$

The time between oscillations is therefore π

c $\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} = -12\sin 2x + 16\sin 2x$

If $\tan x = -\frac{1}{3}$, then

$x = \pi - 0.321\ldots = 2.82$

At $x = 2.82$, $\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} > 0$,

So the displacement is minimised at this point.

4 a $y = \frac{1}{2}\sin\frac{x}{5} + \sin\frac{2x}{5} + \frac{1}{2}\cos\frac{x}{5} + 2$

$\frac{dy}{dx} = \frac{1}{10}\cos\frac{x}{5} + \frac{2}{5}\cos\frac{2x}{5} - \frac{1}{10}\sin\frac{x}{5}$

$$= \frac{1}{10}\cos\frac{x}{5} + \frac{2}{5}\left(\cos^2\frac{x}{5} - \sin^2\frac{x}{5}\right) - \frac{1}{10}\sin\frac{x}{5}$$

Substitute $t = \tan\frac{x}{10}$

$$\frac{dy}{dx} = \frac{1}{10}\left(\frac{1-t^2}{1+t^2}\right) + \frac{2}{5}\left[\frac{\left(1-t^2\right)^2 - (2t)^2}{\left(1+t^2\right)^2}\right] - \frac{1}{10}\left(\frac{2t}{1+t^2}\right)$$

$$= \frac{\left(1-t^2\right)\left(1+t^2\right) + 4\left(\left(1-t^2\right)^2 - 4t^2\right) - 2t\left(1+t^2\right)}{10\left(1+t^2\right)^2}$$

$$= \frac{\left(1-t^2\right)\left(1+t^2\right) + 4\left(1-6t^2+t^4\right) - 2t - 2t^3}{10\left(1+t^2\right)^2}$$

$$= \frac{1-t^4+4-24t^2+4t^4-2t-2t^3}{10\left(1+t^2\right)^2}$$

$$= \frac{3t^4-2t^3-24t^2-2t+5}{10\left(1+t^2\right)^2}$$

$$= \frac{\left(3t^2-8t-5\right)\left(t^2+2t-1\right)}{10\left(1+t^2\right)^2}$$

b **i** Comparing y – values on each graph, $k = \frac{1}{10}$ would be sensible.

ii The model is suitable for predicting times, since both graphs oscillate bi-modally with similar periodicity. However, it is not suitable for predicting intensity, since the peak height is constant for the model, but varies in the observed data.

c Every peak occurs when $\frac{dy}{dx} = 0$ and the most intense peak is the 'fourth occurrence' of the first intense peak.

The first intense peak is the first stationary value, so you are looking for the least value of t satisfying $\frac{dy}{dx} = 0$

So if $3t^2 - 8t - 5 = 0$,

then $t = \frac{8 \pm \sqrt{124}}{6} = \tan\frac{x}{10}$

If $t^2 + 2t - 1 = 0$,

then $t = \frac{-2+\sqrt{8}}{2} = 0.414\ldots = \tan\frac{x}{10}$

The least value of t is 0.414, which corresponds to the first maximum peak.

The general solution to these peaks is $\frac{x}{10} = n\pi + \arctan 0.414$ and the 'fourth occurrence' occurs when $n = 3$

Therefore $\frac{x}{10} = 3\pi + \arctan 0.414$

$x = 10\left(3\pi + \arctan 0.414\right)$

$x = 98$ milliseconds