The *t*– formulae 5C

1 a $2\sin\theta - \cos\theta = 2$

Using the substitution $t = \tan\frac{\theta}{2}$

$2\left(\frac{2t}{1+t^2}\right) - \left(\frac{1-t^2}{1+t^2}\right) = 2$

$4t - 1 + t^2 = 2 + 2t^2$

$t^2 - 4t + 3 = 0$

$(t-3)(t-1) = 0$

$t = 3$ or $t = 1$

$\tan\frac{\theta}{2} = 3$

$\frac{\theta}{2} = 1.249\ldots$

$\theta = 2.50$

or $\tan\frac{\theta}{2} = 1$

$\frac{\theta}{2} = \frac{\pi}{4}$

$\theta = \frac{\pi}{2} = 1.57$

1 b $\sin\theta + 5\cos\theta = -1$

Using the substitution $t = \tan\frac{\theta}{2}$

$\frac{2t}{1+t^2} - 5\left(\frac{1-t^2}{1+t^2}\right) = -1$

$2t + 5 - 5t^2 = -1 - t^2$

$4t^2 - 2t - 6 = 0$

$2t^2 - t - 3 = 0$

$(2t-3)(t+1) = 0$

$t = \frac{3}{2}$ or $t = -1$

$\tan\frac{\theta}{2} = \frac{3}{2}$

$\frac{\theta}{2} = 0.982\ldots$

$\theta = 1.97$

or $\tan\frac{\theta}{2} = -1$

$\frac{\theta}{2} = \frac{3\pi}{4}$

$\theta = \frac{3\pi}{2} = 4.71$

1 c $\tan\theta - 5\sec\theta = 7$

Using the substitution $t = \tan\frac{\theta}{2}$

$\frac{2t}{1-t^2} - 5\left(\frac{1+t^2}{1-t^2}\right) = 7$

$2t - 5 - 5t^2 = 7 - 7t^2$

$2t^2 + 2t - 12 = 0$

$t^2 + t - 6 = 0$

$(t+3)(t-2) = 0$

$t = -3$ or $t = 2$

$\tan\frac{\theta}{2} = -3$

$\frac{\theta}{2} = \pi - 1.249\ldots$

$\theta = 3.79$

or $\tan\frac{\theta}{2} = 2$

$\frac{\theta}{2} = 1.107\ldots$

$\theta = 2.21$

d $7\cot\theta + 3\operatorname{cosec}\theta = 9$

Using the substitution $t = \tan\frac{\theta}{2}$

$7\left(\frac{1-t^2}{2t}\right) + 3\left(\frac{1+t^2}{2t}\right) = 9$

$7 - 7t^2 + 3 + 3t^2 = 18t$

$10 - 4t^2 = 18t$

$4t^2 + 18t - 10 = 0$

$2t^2 + 9t - 5 = 0$

$(2t-1)(t+5) = 0$

$t = \frac{1}{2}$ or $t = -5$

$\tan\frac{\theta}{2} = \frac{1}{2}$

$\frac{\theta}{2} = 0.436\ldots$

$\theta = 0.93$

or $\tan\frac{\theta}{2} = -5$

$\frac{\theta}{2} = \pi - 1.373\ldots$

$\theta = 3.54$

e $2\cot\theta - \operatorname{cosec}\theta = 0$

Using the substitution $t = \tan\frac{\theta}{2}$

$2\left(\frac{1-t^2}{2t}\right) + \left(\frac{1+t^2}{2t}\right) = 0$

$2 - 2t^2 - 1 - t^2 = 0$

$3t^2 - 1 = 0$

$t^2 = \frac{1}{3}$

$t = \frac{1}{\sqrt{3}}$ or $t = -\frac{1}{\sqrt{3}}$

$\tan\frac{\theta}{2} = \frac{1}{\sqrt{3}}$

$\frac{\theta}{2} = \frac{\pi}{6}$

$\theta = \frac{\pi}{3} = 1.05$

or $\tan\frac{\theta}{2} = -\frac{1}{\sqrt{3}}$

$\frac{\theta}{2} = \pi - \frac{\pi}{6}$

$\theta = \frac{5\pi}{3} = 5.24$

2 a $\sin 2\theta - 2\cos 2\theta = 1 - \sqrt{3}\cos 2\theta$

Using the substitution $t = \tan\theta$

$\frac{2t}{1+t^2} - 2\left(\frac{1-t^2}{1+t^2}\right) = 1 - \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$

$2t - 2 + 2t^2 = 1 + t^2 - \sqrt{3} + \sqrt{3}t^2$

$\sqrt{3}t^2 - t^2 - 2t + 3 - \sqrt{3} = 0$

$\left(\sqrt{3} - 1\right)t^2 - 2t - \left(\sqrt{3} - 3\right) = 0$

2 b $t = \dfrac{2 \pm \sqrt{4 + 4(\sqrt{3}-1)(\sqrt{3}-3)}}{2(\sqrt{3}-1)}$

$t = \dfrac{2 \pm \sqrt{28 - 16\sqrt{3}}}{2(\sqrt{3}-1)}$

You need to evaluate $\sqrt{28-16\sqrt{3}}$

Let $a + b\sqrt{3} = \sqrt{28-16\sqrt{3}}$, where a and b are integers.

Then $(a+b\sqrt{3})^2 = 28-16\sqrt{3}$

$a^2 + 3b^2 + 2ab\sqrt{3} = 28 - 16\sqrt{3}$

So $a^2 + 3b^2 = 28$ and $ab = -8$

Solving simultaneously gives:

$a^2 + 3\left(\dfrac{64}{a^2}\right) = 28$

$a^4 + 192 = 28a^2$

$a^4 - 28a^2 + 192 = 0$

$(a^2-16)(a^2-12) = 0$

So $a^2 = 16$ (since a is an integer)

Therefore $a = \pm 4$

$a = 4 \Rightarrow b = -2$

$a = -4 \Rightarrow b = 2$

Therefore $\sqrt{28-16\sqrt{3}} = 4 - 2\sqrt{3}$

or $\sqrt{28-16\sqrt{3}} = -4 + 2\sqrt{3}$

Now $t = \dfrac{2 \pm \sqrt{28-16\sqrt{3}}}{2(\sqrt{3}-1)}$

So $t = \dfrac{2 + \sqrt{28-16\sqrt{3}}}{2(\sqrt{3}-1)}$

$= \dfrac{2 + (4 - 2\sqrt{3})}{2(\sqrt{3}-1)} = \dfrac{6 - 2\sqrt{3}}{2\sqrt{3} - 2}$

$= \dfrac{\sqrt{3}(2\sqrt{3}-2)}{2\sqrt{3}-2}$

$= \sqrt{3}$

or $t = \dfrac{2 + (-4 + 2\sqrt{3})}{2(\sqrt{3}-1)} = \dfrac{-2 + 2\sqrt{3}}{2\sqrt{3}-2} = 1$

Therefore $t = \sqrt{3} \Rightarrow \theta = \dfrac{\pi}{3}$ or $\theta = \dfrac{4\pi}{3}$

and $t = 1 \Rightarrow \theta = \dfrac{\pi}{4}$ or $\theta = \dfrac{5\pi}{4}$

3 a $16\cot x - 9\tan x = 0$

Using the substitution $t = \tan\dfrac{x}{2}$

$16\left(\dfrac{1-t^2}{2t}\right) - 9\left(\dfrac{2t}{1-t^2}\right) = 0$

$16(1-t^2)^2 - 18t(2t) = 0$

$16(1 - 2t^2 + t^4) - 36t^2 = 0$

$16 - 32t^2 + 16t^4 - 36t^2 = 0$

$16t^4 - 68t^2 + 16 = 0$

$4t^4 - 17t^2 + 4 = 0$

b Substituting $u = t^2$

$4u^2 - 17u + 4 = 0$

$(4u-1)(u-4) = 0$

$u = \dfrac{1}{4}$ or $u = 4$

$t^2 = \dfrac{1}{4}$ or $t^2 = 4$

$t = \dfrac{1}{2}$ or $t = -\dfrac{1}{2}$ or $t = 2$ or $t = -2$

$\tan\dfrac{\theta}{2} = \dfrac{1}{2}$

$\dfrac{\theta}{2} = 0.463\ldots$

$\theta = 0.93$

$\tan\dfrac{\theta}{2} = -\dfrac{1}{2}$

$\dfrac{\theta}{2} = \pi - 0.463\ldots$

$\theta = 5.36$

$\tan\dfrac{\theta}{2} = 2$

$\dfrac{\theta}{2} = 1.107\ldots$

$\theta = 2.21$

$\tan\dfrac{\theta}{2} = -2$

$\dfrac{\theta}{2} = \pi - 1.107\ldots$

$\theta = 4.07$

4 a $10\sin\theta\cos\theta - 3\cos\theta = -3$

Using the substitution $t = \tan\frac{x}{2}$

$10\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{1-t^2}{1+t^2}\right) = -3$

$20t\left(1-t^2\right) - 3\left(1-t^4\right) = -3\left(1+t^2\right)^2$

$20t - 20t^3 - 3 + 3t^4 = -3\left(1+2t^2+t^4\right)$

$20t - 20t^3 - 3 + 3t^4 = -3 - 6t^2 - 3t^4$

$6t^4 - 20t^3 + 6t^2 + 20t = 0$

$3t^4 - 10t^3 + 3t^2 + 10t = 0$

$t\left(3t^3 - 10t^2 + 3t + 10\right) = 0$

Let $f(t) = 3t^3 - 10t^2 + 3t + 10$

Then $f(2) = 3\times2^3 - 10\times2^2 + 6 + 10 = 0$

So $(t-2)$ is a factor of $f(t)$

$3t^3 - 10t^2 + 3t + 10 = (t-2)\left(3t^2 + kt - 5\right)$

Equating coefficients of t^2:

$-10 = k - 6$

So $k = -4$

$\therefore 3t^3 - 10t^2 + 3t + 10 = (t-2)\left(3t^2 - 4t - 5\right)$

$\therefore t(t-2)\left(3t^2 - 4t - 5\right) = 0$

b So $t = 0$, $t = 2$, or $t = \frac{4\pm\sqrt{76}}{6} = \frac{2\pm\sqrt{19}}{3}$

$\tan\frac{\theta}{2} = 0$

$\frac{\theta}{2} = 0$

$\theta = 0$

$\tan\frac{\theta}{2} = 2$

$\frac{\theta}{2} = 1.107\ldots$

$\theta = 2.21$

$\tan\frac{\theta}{2} = \frac{2+\sqrt{19}}{3}$

$\frac{\theta}{2} = 1.129\ldots$

$\theta = 2.26$

$\tan\frac{\theta}{2} = \frac{2-\sqrt{19}}{3}$

$\frac{\theta}{2} = \pi - 0.666\ldots$

$\theta = 4.95$

5 a $3\sin2\theta + \cos2\theta + 3\tan2\theta = 1$

Using the substitution $t = \tan\theta$

$3\left(\frac{2t}{1+t^2}\right) + \frac{1-t^2}{1+t^2} + 3\left(\frac{2t}{1-t^2}\right) = 1$

$6t\left(1-t^2\right) + \left(1-t^2\right)^2 + 6t\left(1+t^2\right)$

$= \left(1+t^2\right)\left(1-t^2\right)$

$6t - 6t^3 + 1 - 2t^2 + t^4 + 6t + 6t^3 = 1 - t^4$

$2t^4 - 2t^2 + 12t = 0$

$t^4 - t^2 + 6t = 0$

b $t^4 - t^2 + 6t = t\left(t^3 - t + 6\right)$

$(t+2)$ is a factor of $t^3 - t + 6$,

So $t^3 - t + 6 = (t+2)\left(t^2 + kt + 3\right)$

Equating coefficients of t^2:

$0 = 2 + k$

So $k = -2$

Therefore $t(t+2)\left(t^2 - 2t + 3\right) = 0$

Note that $t^2 - 2t + 3 = 0$ has no solutions, since

$'b^2 - 4ac' = (-2)^2 - 4\times1\times3 = -8 < 0$

Therefore $t = 0$ or $t = -2$

$\tan\theta = 0$

$\theta = 0$ or $\theta = \pi$

$\tan\theta = -2$

$\theta = \pi - 1.107\ldots$ or $\theta = 2\pi - 1.107\ldots$

$\theta = 2.03$ or $\theta = 5.18$

6 a $\tan\theta + \cos2\theta = 1$

Using the substitution $t = \tan\theta$

$t + \left(\frac{1-t^2}{1+t^2}\right) = 1$

$t\left(1+t^2\right) + 1 - t^2 = 1 + t^2$

$t + t^3 - t^2 = t^2$

$t^3 - 2t^2 + t = 0$

6 b $t(t^2-2t+1)=0$

$t(t-1)^2=0$

$\tan\theta=0$
$\theta=0$ or $\theta=\pi$ or $\theta=2\pi$

$\tan\theta=1$

$\theta=\frac{\pi}{4}$ or $\theta=\pi+\frac{\pi}{4}=\frac{5\pi}{4}$

7 a $2\sin 2\theta-\cos 4\theta-4\tan\theta=-1$

$2\sin 2\theta-(\cos^2 2\theta-\sin^2 2\theta)-4\tan\theta$
$=-1$

$2\sin 2\theta-\cos^2 2\theta+\sin^2 2\theta-4\tan\theta$
$=-1$

Using the substitution $t=\tan\theta$

$$2\left(\frac{2t}{1+t^2}\right)-\left(\frac{1-t^2}{1+t^2}\right)^2+\left(\frac{2t}{1+t^2}\right)^2-4t=-1$$

$$\frac{4t(1+t^2)-(1-t^2)^2+4t^2-4t(1+t^2)^2}{(1+t^2)^2}$$
$=-1$

$4t(1+t^2)-(1-t^2)^2+4t^2-4t(1+t^2)^2$
$=-(1+t^2)^2$

$4t+4t^3-(1-2t^2+t^4)+4t^2-4t(1+2t^2+t^4)$
$=-(1+2t^2+t^4)$

$4t+4t^3-1+2t^2-t^4+4t^2-4t-8t^3-4t^5$
$=-1-2t^2-t^4$

$-4t^5-4t^3+8t^2=0$

$t^5+t^3-2t^2=0$

b $t^5+t^3-2t^2=0$

$t^2(t^3+t-2)=0$

Let $f(t)=t^3+t-2$

$f(1)=1^3+1-2=0$

So $t-1$ is a factor of $f(t)$

So $f(t)=(t-1)(t^2+kt+2)$

Equating coefficients of t^2
$0=-1+k$
$k=1$
Therefore $t^2(t-1)(t^2+t+2)=0$

Note that $t^2+t+2=0$
has no solutions, since
'b^2-4ac' $=1^2-4\times1\times2=-7<0$

So $t=0$ or $t=1$

$\tan\theta=0$
$\theta=0$, $\theta=\pi$, $\theta=2\pi$

$\tan\theta=1$

$\theta=\frac{\pi}{4}$, $\theta=\pi+\frac{\pi}{4}=\frac{5\pi}{4}$

8 $5\cos\theta - 12\operatorname{cosec}\theta = 12$

Using the substitution $t = \tan\frac{\theta}{2}$

$5\left(\frac{1-t^2}{1+t^2}\right) - 12\left(\frac{1+t^2}{2t}\right) = 12$

$5(2t)(1-t^2) - 12(1+t^2)^2 = 12(2t)(1+t^2)$

$10t(1-t^2) - 12(1+2t^2+t^4) = 24t + 24t^3$

$10t - 10t^3 - 12 - 24t^2 - 12t^4$

$= 24t + 24t^3$

$12t^4 + 34t^3 + 24t^2 + 14t + 12 = 0$

$6t^4 + 17t^3 + 12t^2 + 7t + 6 = 0$

Let $f(t) = 6t^4 + 17t^3 + 12t^2 + 7t + 6$

$f(-1) = 6 - 17 + 12 - 7 + 6 = 0$

So $(t+1)$ is a factor of $f(t)$

Also $f(-2) = 96 - 136 + 48 - 14 + 6 = 0$

So $(t+2)$ is a factor of $f(t)$

Therefore $(t+1)(t+2) = t^2 + 3t + 2$

is a factor of $f(t)$

$6t^4 + 17t^3 + 12t^2 + 7t + 6$

$= (t^2 + 3t + 2)(6t^2 + kt + 3)$

Equate coefficients of t^2:

$12 = 3 + 3k + 12$

So $k = -1$

So $(t+1)(t+2)(6t^2 - t + 3) = 0$

Note that $6t^2 - t + 3 = 0$

has no solutions, since

$'b^2 - 4ac' = (-1)^2 - 4\times 6\times 3 = -71 < 0$

So $t = -1$ or $t = -2$

$\tan\frac{\theta}{2} = -1$

$\frac{\theta}{2} = \pi - \frac{\pi}{4}$

$\theta = \frac{3\pi}{2} = 4.71$

$\tan\frac{\theta}{2} = -2$

$\frac{\theta}{2} = \pi - 1.107\ldots$

$\theta = 4.07$

Challenge

$5\sin 2\theta + 12\cos\theta = -12$

$10\sin\theta\cos\theta + 12\cos\theta = -12$

Substitute $t = \tan\frac{\theta}{2}$

$10\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right) + 12\left(\frac{1-t^2}{1+t^2}\right) = 12$

$\frac{20t(1-t^2)}{(1+t^2)^2} + \frac{12(1-t^2)}{1+t^2} = -12$

$20t(1-t^2) + 12(1-t^2)(1+t^2) = -12(1+t^2)^2$

$20t - 20t^3 + 12(1-t^4) = -12(1+2t^2+t^4)$

$20t - 20t^3 + 12 - 12t^4 = -12 - 24t^2 - 12t^4$

$20t^3 - 20t - 24t^2 - 24 = 0$

$5t^3 - 6t^2 - 5t - 6 = 0$

Let $f(t) = 5t^3 - 6t^2 - 5t - 6$

$f(2) = 5\times 2^3 - 6\times 2^2 - 5\times 2 - 6 = 0$

So $t - 2$ is a factor of $f(t)$

So $f(t) = (t-2)(5t^2 + kt + 3)$

Equating coefficients of t^2

$-6 = -10 + k$

$k = 4$

Therefore $(t-2)(5t^2 + 4t + 3)$

Note that $5t^2 + 4t + 3 = 0$

has no solutions, since

$'b^2 - 4ac' = 4^2 - 4\times 5\times 3 = -34 < 0$

So $t = 2$

$\tan\frac{\theta}{2} = 2$

$\frac{\theta}{2} = 1.107\ldots,$

$\theta = 2.21$

Check possible values of θ

for which $\tan\left(\frac{\theta}{2}\right)$ is undefined,

and which is in the given range $0 \leqslant \theta \leqslant 2\pi$

ie. $\theta = \pi$

This gives

$5\sin 2\pi + 12\cos\pi = 5\times 0 + 12\times(-1) = -12$

So $\theta = \pi$ is also a solution.