## The *t*– formulae 5B

**1 a** $\tan\frac{\theta}{2}=t$, so

$$\sin^2\theta+\cos^2\theta=\left(\frac{2t}{1+t^2}\right)^2+\left(\frac{1-t^2}{1+t^2}\right)^2$$
$$=\frac{4t^2+1-2t^2+t^4}{\left(1+t^2\right)^2}=\frac{t^4+2t^2+1}{\left(1+t^2\right)^2}$$
$$=\frac{\left(1+t^2\right)^2}{\left(1+t^2\right)^2}=1$$

**b**
$$\frac{\tan^2\theta}{\tan^2\theta+1}=\frac{\left(\frac{2t}{1-t^2}\right)^2}{\left(\frac{2t}{1-t^2}\right)^2+1}=\frac{\frac{4t^2}{\left(1-t^2\right)^2}}{\frac{4t^2}{\left(1-t^2\right)^2}+1}$$
$$=\frac{\frac{4t^2}{\left(1-t^2\right)^2}}{\frac{4t^2+\left(1-t^2\right)^2}{\left(1-t^2\right)^2}}=\frac{4t^2}{4t^2+\left(1-t^2\right)^2}$$
$$=\frac{4t^2}{4t^2+t^4-2t^2+1}=\frac{4t^2}{t^4+2t^2+1}$$
$$=\frac{4t^2}{\left(1+t^2\right)^2}=\left(\frac{2t}{1+t^2}\right)^2=\sin^2\theta$$

**c**
$$\frac{\operatorname{cosec}\theta}{\sin\theta}-\frac{\cot\theta}{\tan\theta}=\frac{1}{\sin^2\theta}-\frac{1}{\tan^2\theta}$$
$$=\frac{\left(1+t^2\right)^2}{4t^2}-\frac{\left(1-t^2\right)^2}{4t^2}$$
$$=\frac{1+2t^2+t^4-\left(1-2t^2+t^4\right)}{4t^2}$$
$$=\frac{4t^2}{4t^2}=1$$

**d**
$$\cot 2\theta+\tan\theta=\frac{1}{\tan 2\theta}+\tan\theta$$
$$\frac{1-t^2}{2t}+t=\frac{1-t^2+2t^2}{2t}$$
$$=\frac{1+t^2}{2t}=\frac{1}{\sin 2\theta}=\operatorname{cosec}2\theta$$

**2 a**
$$\tan\theta+\cot\theta=\tan\theta+\frac{1}{\tan\theta}$$
$$=\frac{2t}{1-t^2}+\frac{1-t^2}{2t}=\frac{4t^2+\left(1-t^2\right)^2}{2t\left(1-t^2\right)}$$
$$=\frac{4t^2+1-2t^2+t^4}{2t\left(1-t^2\right)}=\frac{t^4+2t^2+1}{2t\left(1-t^2\right)}$$
$$=\frac{\left(1+t^2\right)^2}{2t\left(1-t^2\right)}$$
$$=\frac{1+t^2}{1-t^2}\times\frac{1+t^2}{2t}$$
$$=\sec\theta\operatorname{cosec}\theta$$

**b**
$$\frac{1+\cos\theta}{\sin\theta}=\frac{1+\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}}$$
$$=\frac{1+t^2+1-t^2}{2t}=\frac{2}{2t}=\frac{1}{t}$$
$$=\frac{2t}{2t^2}=\frac{2t}{1+t^2-\left(1-t^2\right)}$$
$$=\frac{\frac{2t}{1+t^2}}{1-\left(\frac{1-t^2}{1+t^2}\right)}=\frac{\sin\theta}{1-\cos\theta}$$

**c**
$$\frac{1-\sin\theta}{\cos\theta}=\frac{1-\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}}$$
$$=\frac{1+t^2-2t}{1-t^2}=\frac{\left(1-t\right)^2}{\left(1+t\right)\left(1-t\right)}=\frac{1-t}{1+t}$$
$$=\frac{\left(1-t\right)\left(1+t\right)}{\left(1+t\right)^2}=\frac{1-t^2}{\left(1+t\right)^2}$$
$$=\frac{\frac{1-t^2}{1+t^2}}{\frac{\left(1+t\right)^2}{1+t^2}}=\frac{\frac{1-t^2}{1+t^2}}{\frac{1+t^2+2t}{1+t^2}}$$
$$=\frac{\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}}=\frac{\cos\theta}{1+\sin\theta}$$

**2 d** $\tan\theta\sin\theta+\cos\theta=\frac{2t}{1-t^2}\left(\frac{2t}{1+t^2}\right)+\frac{1-t^2}{1+t^2}$

$=\frac{4t^2}{1-t^4}+\frac{1-t^2}{1+t^2}=\frac{4t^2+\left(1-t^2\right)^2}{1-t^4}$

$=\frac{4t^2+1-2t^2+t^4}{1-t^4}=\frac{t^4+2t^2+1}{1-t^4}$

$=\frac{\left(1+t^2\right)\left(1+t^2\right)}{\left(1+t^2\right)\left(1-t^2\right)}=\frac{1+t^2}{1-t^2}$

$=\frac{1}{\cos\theta}=\sec\theta$

**3** $\sin\theta+\sin\theta\cot^2\theta$

$=\frac{2t}{1+t^2}+\frac{2t}{1+t^2}\left(\frac{\left(1-t^2\right)^2}{4t^2}\right)$

$=\frac{2t}{1+t^2}+\frac{\left(1-t^2\right)^2}{2t\left(1+t^2\right)}$

$=\frac{4t^2+\left(1-t^2\right)^2}{2t\left(1+t^2\right)}=\frac{4t^2+t^4-2t^2+1}{2t\left(1+t^2\right)}$

$=\frac{t^4+2t^2+1}{2t\left(1+t^2\right)}=\frac{\left(1+t^2\right)^2}{2t\left(1+t^2\right)}$

$=\frac{1+t^2}{2t}=\frac{1}{\sin\theta}=\operatorname{cosec}\theta$

**4** $\frac{\cos\theta}{1-\sin\theta}-\frac{\cos\theta}{1+\sin\theta}=\frac{\frac{1-t^2}{1+t^2}}{1-\frac{2t}{1+t^2}}-\frac{\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}}$

$=\frac{1-t^2}{1+t^2-2t}-\frac{1-t^2}{1+t^2+2t}$

$=\frac{(1+t)(1-t)}{(1-t)^2}-\frac{(1+t)(1-t)}{(1+t)^2}$

$=\frac{(1+t)}{(1-t)}-\frac{(1-t)}{(1+t)}$

$=\frac{(1+t)^2-(1-t)^2}{1-t^2}$

$=\frac{1+2t+t^2-\left(1-2t+t^2\right)}{1-t^2}$

$=\frac{4t}{1-t^2}=2\times\frac{2t}{1-t^2}$

$=2\tan\theta$

**5** $\frac{\operatorname{cosec}x\cos x}{\tan x+\cot x}=\frac{\cot x}{\tan x+\cot x}$

$=\frac{\frac{1-t^2}{2t}}{\frac{2t}{1-t^2}+\frac{1-t^2}{2t}}=\frac{\left(1-t^2\right)^2}{4t^2+\left(1-t^2\right)^2}$

$=\frac{\left(1-t^2\right)^2}{\left(1+t^2\right)^2}=\left(\frac{1-t^2}{1+t^2}\right)^2$

$=\cos^2 x$

**6** $\frac{\cos\theta}{1+\sin\theta}+\frac{1+\sin\theta}{\cos\theta}=\frac{\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}}+\frac{1+\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}}$

$=\frac{1-t^2}{1+t^2+2t}+\frac{1+t^2+2t}{1-t^2}$

$=\frac{(1+t)(1-t)}{(1+t)^2}+\frac{(1+t)^2}{(1+t)(1-t)}$

$=\frac{(1-t)}{(1+t)}+\frac{(1+t)}{(1-t)}$

$=\frac{(1-t)^2+(1+t)^2}{1-t^2}$

$=\frac{1-2t+t^2+\left(1+2t+t^2\right)}{1-t^2}$

$=\frac{2+2t^2}{1-t^2}=\frac{2\left(1+t^2\right)}{1-t^2}$

$=\frac{2}{\cos\theta}=2\sec\theta$

**7** $\sec\theta+\tan\theta=\frac{1}{\cos\theta}+\tan\theta$

$=\frac{1+t^2}{1-t^2}+\frac{2t}{1-t^2}$

$=\frac{1+t^2+2t}{1-t^2}=\frac{(1+t)^2}{(1+t)(1-t)}=\frac{1+t}{1-t}$

$=\left(\frac{1+t}{1-t}\right)\left(\frac{1-t}{1-t}\right)=\frac{1-t^2}{1+t^2-2t}$

$=\frac{\frac{1-t^2}{1+t^2}}{1-\frac{2t}{1+t^2}}=\frac{\cos\theta}{1-\sin\theta}$

**8**
$$\frac{1+\sin 2x-\cos 2x}{\sin 2x+\cos 2x-1}$$
$$=\frac{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}-1}$$
$$=\frac{1+t^2+2t-1+t^2}{2t+1-t^2-1-t^2}$$
$$=\frac{2t+2t^2}{2t-2t^2}$$
$$=\frac{2t(1+t)}{2t(1-t)}$$
$$=\frac{1+t}{1-t}$$
$$=\frac{1+\tan x}{1-\tan x}$$

**9**
$$\frac{\cos\theta}{1-\sin\theta}-\tan\theta=\frac{\frac{1-t^2}{1+t^2}}{1-\frac{2t}{1+t^2}}-\frac{2t}{1-t^2}$$
$$=\frac{1-t^2}{1+t^2-2t}-\frac{2t}{1-t^2}$$
$$=\frac{1-t^2}{(1-t)^2}-\frac{2t}{(1+t)(1-t)}$$
$$=\frac{(1+t)(1-t)}{(1-t)^2}-\frac{2t}{(1+t)(1-t)}$$
$$=\frac{1+t}{1-t}-\frac{2t}{(1+t)(1-t)}$$
$$=\frac{(1+t)^2-2t}{(1+t)(1-t)}$$
$$=\frac{1+t^2+2t-2t}{(1+t)(1-t)}$$
$$=\frac{1+t^2}{1-t^2}=\frac{1}{\cos\theta}=\sec\theta$$

**10**
$$\tan^2\theta+\tan\theta\sec\theta+1$$
$$=\left(\frac{2t}{1-t^2}\right)^2+\left(\frac{2t}{1-t^2}\right)\left(\frac{1+t^2}{1-t^2}\right)+1$$
$$=\frac{4t^2}{(1-t^2)^2}+\frac{2t(1+t^2)}{(1-t^2)^2}+\frac{(1-t^2)^2}{(1-t^2)^2}$$
$$=\frac{4t^2+2t+2t^3+1-2t^2+t^4}{(1-t^2)^2}$$
$$=\frac{2t^2+2t+2t^3+1+t^4}{(1-t^2)^2}$$
$$=\frac{(1+t^2)^2+2t(1+t^2)}{(1-t^2)^2}$$
$$=\frac{1+\frac{2t}{1+t^2}}{\frac{(1-t^2)^2}{(1+t^2)^2}}$$
$$=\frac{1+\frac{2t}{1+t^2}}{\left(\frac{1-t^2}{1+t^2}\right)^2}$$
$$=\frac{1+\sin\theta}{\cos^2\theta}$$

**11**
$$\text{LHS}=\frac{\cos 2x}{1-\sin 2x}=\frac{\frac{1-t^2}{1+t^2}}{1-\frac{2t}{1+t^2}}$$
$$=\frac{1-t^2}{1+t^2-2t}=\frac{(1+t)(1-t)}{(1-t)^2}$$
$$=\frac{1+t}{1-t}$$
$$\text{RHS}=\frac{\cot x+1}{\cot x-1}=\frac{\frac{1}{t}+1}{\frac{1}{t}-1}=\frac{1+t}{1-t}$$
Therefore $\dfrac{\cos 2x}{1-\sin 2x}=\dfrac{\cot x+1}{\cot x-1}$

**Challenge**

$$\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}=\frac{\left(\frac{2t}{1+t^2}\right)^3+\left(\frac{1-t^2}{1+t^2}\right)^3}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}$$

$$=\frac{\frac{(2t)^3}{(1+t^2)^2}+\frac{(1-t^2)^3}{(1+t^2)^2}}{1+2t-t^2}$$

$$=\frac{(2t)^3+(1-t^2)^3}{(1+t^2)^2(1+2t-t^2)}$$

$$=\frac{8t^3+1-3t^2+3t^4-t^6}{(1+t^2)^2(1+2t-t^2)}$$

$$=\frac{1-3t^2+8t^3+3t^4-t^6}{(1+t^2)^2(1+2t-t^2)} \qquad \textbf{(1)}$$

Also $1-\sin\theta\cos\theta=1-\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)$

$$=\frac{(1+t^2)^2-2t(1-t^2)}{(1+t^2)^2}$$

$$=\frac{1+2t^2+t^4-2t+2t^3}{(1+t^2)^2}$$

$$=\frac{1-2t+2t^2+2t^3+t^4}{(1+t^2)^2} \qquad \textbf{(2)}$$

To show expressions **(1)** and **(2)** are identical, it remains to check:

$$(1-2t+2t^2+2t^3+t^4)(1+2t-t^2)$$

$$=1-2t+2t^2+2t^3+t^4$$
$$+2t-4t^2+4t^3+4t^4+2t^5$$
$$-t^2+2t^3-2t^4-2t^5-t^6$$

$=1-3t^2+8t^3+3t^4-t^6$, as required.

Therefore $\dfrac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}=1-\sin\theta\cos\theta$