Conic sections 2 3F

1 a Using the table in Section 3.6

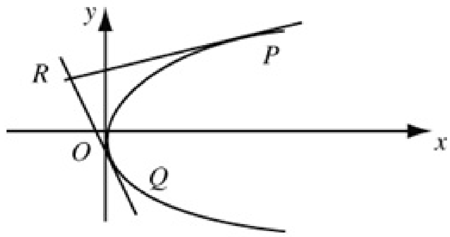
Tangent at P is $py = x + ap^2$

Tangent at Q is $qy = x + aq^2$

$(p-q)y = a(p-q)(p+q) \quad \therefore y = a(p+q)$

$\Rightarrow ap^2 + apq = x + ap^2 \quad \therefore x = apq$

So R is $(apq,\ a(p+q))$



b Chord PQ has gradient: $\dfrac{2ap-2aq}{ap^2-aq^2} = \dfrac{2a(p-q)}{a(p-q)(p+q)} = \dfrac{2}{p+q}$

Equation of chord PQ is: $y - 2ap = \dfrac{2}{p+q}(x - ap^2)$

$\Rightarrow y(p+q) - \cancel{2ap^2} - 2apq = 2x - \cancel{2ap^2}$

$\Rightarrow y(p+q) = 2x + 2apq$

Chord passes through $(a,\ 0) \Rightarrow 0 = 2a + 2apq$ or $pq = -1$

Locus of R is $x = -a$

c Gradient of chord PQ is $\dfrac{2}{p+q} = 2 \Rightarrow p + q = 1$

So locus of R is: $y = a(p+q) = a$

$y = a$

2 a Use the chain rule to find the gradient: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{b\sec^2 t}{a\sec t\tan t} = \dfrac{b}{a\sin t}$

The equation of the tangent l_1 is: $y - b\tan t = \dfrac{b}{a\sin t}(x - a\sec t)$

$$ay\sin t - ab\sin t\tan t = bx - ab\sec t$$

$$ay\tan t - ab\tan^2 t = bx\sec t - ab\sec^2 t$$

$$bx\sec t - ay\tan t = ab(\sec^2 t - \tan^2 t) = ab$$

b $x = 0$: $-ay\tan t = ab$, so $y = -\dfrac{b}{\tan t}$

$y = 0$: $bx\sec t = ab$, so $x = \dfrac{a}{\sec t}$

So the point A has coordinates $\left(0, -\dfrac{b}{\tan t}\right)$, while B has coordinates $\left(\dfrac{a}{\sec t}, 0\right)$

The midpoint of AB has coordinates (X, Y) where $X = \dfrac{a}{2\sec t}$ and $Y = -\dfrac{b}{2\tan t}$

Rearranging: $\sec t = \dfrac{a}{2X}$ and $\tan t = -\dfrac{b}{2Y}$

Using $\sec^2 t - \tan^2 t \equiv 1$ gives the locus: $\dfrac{a^2}{4X^2} - \dfrac{b^2}{4Y^2} = 1$

3 a Use the chain rule to find the gradient of the tangent:

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{b\sec^2 t}{a\tan t\sec t}=\frac{b}{a\sin t}$$

Then the normal has gradient $-\frac{a\sin t}{b}$

An equation of the normal l_1 is: $y-b\tan t=-\frac{a\sin t}{b}(x-a\sec t)$

$$by-b^2\tan t=-ax\sin t+a^2\tan t$$

$$ax\sin t+by=(a^2+b^2)\tan t$$

b $x=0$: $by=(a^2+b^2)\tan t$, so $y=\frac{a^2+b^2}{b}\tan t$

$y=0$: $ax\sin t=(a^2+b^2)\tan t$, so $x=\frac{a^2+b^2}{a}\sec t$

So the point A has coordinates $\left(\frac{a^2+b^2}{a}\sec t,0\right)$, while B has coordinates $\left(0,\frac{a^2+b^2}{b}\tan t\right)$

The midpoint of AB has coordinates (X, Y) where $X=\frac{a^2+b^2}{2a}\sec t$ and $Y=\frac{a^2+b^2}{2b}\tan t$

Rearranging: $\sec t=\frac{2aX}{a^2+b^2}$ and $\tan t=\frac{2bY}{a^2+b^2}$

Using $\sec^2 t-\tan^2 t\equiv 1$ gives the locus: $\frac{4a^2X^2}{(a^2+b^2)^2}-\frac{4b^2Y^2}{(a^2+b^2)^2}=1$

Rearranging into the specified form: $4a^2X^2=(a^2+b^2)^2+4b^2Y^2$

4 a Use the chain rule to find the gradient of the tangent: $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{3\cos\theta}{-5\sin\theta}$

The gradient of the normal l_1 is $\frac{5\sin\theta}{3\cos\theta}$, and an equation for the normal is given by

$$y-3\sin\theta=\frac{5\sin\theta}{3\cos\theta}(x-5\cos\theta)$$

$$\Rightarrow 3y\cos\theta-9\sin\theta\cos\theta=5x\sin\theta-25\sin\theta\cos\theta$$

$$3y\cos\theta=5x\sin\theta-16\sin\theta\cos\theta$$

4 b $x = 0$: $3y = -16\sin\theta$, so $y = -\frac{16}{3}\sin\theta$

$y = 0$: $5x = 16\cos\theta$, so $x = \frac{16}{5}\cos\theta$

So M has coordinates $\left(\frac{16}{5}\cos\theta, 0\right)$, while N has coordinates $(0, -\frac{16}{3}\sin\theta)$

The midpoint of MN has coordinates (X, Y) where $X = \frac{8}{5}\cos\theta$ and $Y = -\frac{8}{3}\sin\theta$

Rearranging: $\cos\theta = \frac{5}{8}X$ and $\sin\theta = -\frac{3}{8}Y$

Using $\cos^2\theta + \sin^2\theta \equiv 1$ gives the locus as $\frac{25}{64}X^2 + \frac{9}{64}Y^2 = 1$

5 a From the table in Section 3.6, the equation of tangent at P is:

$x + p^2 y = 2cp$

Similarly the equation of tangent at Q is:

$x + q^2 y = 2cq$

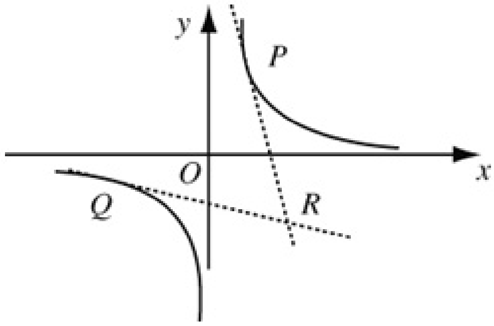
At R the lines intersect, so:

$p^2 y - q^2 y = 2cp - 2cq$

Solving: $\cancel{(p-q)}(p+q)y = 2c\cancel{(p-q)}$

$$\Rightarrow y = \frac{2c}{p+q},\ x = \frac{2cpq}{p+q}$$

So R is $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$



b Gradient of chord PQ is: $\dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \dfrac{c(q-p)}{pqc(p-q)} = -\dfrac{1}{pq}$

So the equation of chord is: $y - \dfrac{c}{p} = -\dfrac{1}{pq}(x - cp)$

$$\Rightarrow ypq - cq = cp - x$$

$$ypq + x = c(p+q)$$

c i $-\dfrac{1}{pq} = 2 \Rightarrow pq = -\dfrac{1}{2}$

R is: $x = -\dfrac{c}{p+q},\ y = \dfrac{2c}{p+q} \Rightarrow y = -2x \ \ (x \neq 0)$

ii Chord passes through $(1, 0) \Rightarrow 1 = c(p+q)$

R is $x = \dfrac{2cpq}{\frac{1}{c}},\ y = \dfrac{2c}{\frac{1}{c}} \Rightarrow y = 2c^2 \ \ (x < 0)$

iii Chord passes through $(0, 1) \Rightarrow pq = c(p+q)$

R is $x = \dfrac{2c^2(p+q)}{(p+q)} \Rightarrow x = 2c^2$

6 a $\left.\begin{array}{l} y = 2at \\ x = at^2 \end{array}\right\} \Rightarrow \dfrac{dy}{dx} = \dfrac{2a}{2at} = \dfrac{1}{t}$

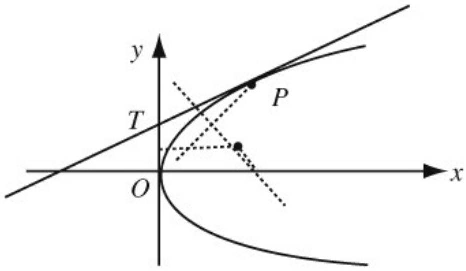
b Equation of tangent is: $y - 2at = \dfrac{1}{t}(x - at^2)$

$$yt - 2at^2 = x - at^2$$

$$yt = x + at^2$$

So $$x - ty + at^2 = 0$$

c T is $(0, at)$
Centre of circle will be intersection of perpendicular bisectors of OT and OP.



Equation of the perpendicular bisector of OT is:
$y = \dfrac{at}{2}$

Midpoint of OP is $\left(\dfrac{at^2}{2}, at\right)$

Gradient of $OP = \dfrac{2at}{at^2} = \dfrac{2}{t}$

So the equation of the perpendicular bisector of OP is: $y - at = -\dfrac{t}{2}\left(x - \dfrac{at^2}{2}\right)$

Perpendicular bisectors intersect when $y = \dfrac{at}{2}$:

$\dfrac{at}{2} - at = -\dfrac{t}{2}\left(x - \dfrac{at^2}{2}\right) \Rightarrow x = \dfrac{at^2}{2} + a$

So the centre of the circle is $\left(\dfrac{at^2}{2} + a, \dfrac{at}{2}\right)$

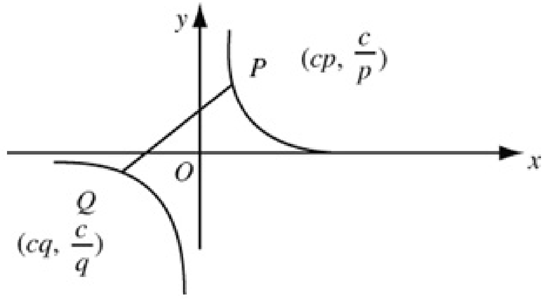
d Let the coordinates of the centre of the circle be (X, Y).

$X = a + \dfrac{at^2}{2} \Rightarrow at^2 = 2(X - a)$

$Y = \dfrac{at}{2} \Rightarrow 2at = 4Y$

So $(4Y)^2 = 4a \times 2(X - a)$ or $2Y^2 = a(X - a)$, which is the equation of a parabola.

7 Using the results from the table in Section 3.6 for the general points on a hyperbola, P is $\left(c, \dfrac{c}{p}\right)$ and Q is $\left(c, \dfrac{c}{q}\right)$



Gradient of chord PQ: $\dfrac{\frac{c}{p}-\frac{c}{q}}{cp-cq}=\dfrac{c(q-p)}{pqc(p-q)}=-\dfrac{1}{pq}$

So the equation of the chord is: $y-\dfrac{c}{p}=-\dfrac{1}{pq}(x-cp)$

$$\Rightarrow ypq-cq=cp-x$$

$$ypq+x=c(p+q)$$

Midpoint of chord PQ is $\left(\dfrac{c(p+q)}{2}, \dfrac{c(p+q)}{2pq}\right)$

Chord passes through $(0, 1) \Rightarrow pq=c(p+q)$

Midpoint is: $x=\dfrac{c(p+q)}{2}, \quad y=\dfrac{2(p+q)}{2pq}$

Substitute $pq=c(p+q) \Rightarrow y=\dfrac{c(p+q)}{2c(p+q)}=\dfrac{1}{2}$

So the locus is the line $y=\dfrac{1}{2}\ (x<0)$

8 a Let $P=(x_1, y_1)$

Since N lies on the line $y = 6$, N has coordinates $(x_1, 6)$

M is the midpoint of PN, so the coordinates of M are $\left(x_1, \dfrac{y_1+6}{2}\right)$

So if M is (x, y), then $x=x_1$ and $y=\dfrac{y_1+6}{2} \Rightarrow y_1=2y-6$

Since P lies on the ellipse, the coordinates $P=(x_1, y_1)$ satisfy the equation of the ellipse.

So M always satisfies the equation $\dfrac{x^2}{4}+\dfrac{(2y-6)^2}{16}=1$

Rearranging: $x^2+y^2-6y+5=0$

b The equation describes a circle, because x^2 and y^2 have the same coefficient.

Rewrite as: $x^2+y^2-6y+9=4$

which leads to $x^2+(y-3)^2=4$

This is the equation of a circle with centre (0, 3) and radius 2

Challenge

Let $A=(x_1, y_1)$ and $B=(x_2, y_2)$

The chord AB has gradient k, so $k=\frac{y_2-y_1}{x_2-x_1}$ and the midpoint (x, y) of AB is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Since A and B are both on the ellipse, their coordinates satisfy the equation of the ellipse.

$$\left.\begin{aligned} b^2x_1^2+a^2y_1^2&=a^2b^2 \\ b^2x_2^2+a^2y_2^2&=a^2b^2 \end{aligned}\right\} \Rightarrow a^2(y_2^2-y_1^2)+b^2(x_2^2-x_1^2)=0$$

$$\Rightarrow a^2(y_2-y_1)(y_2+y_1)=-b^2(x_2-x_1)(x_2+x_1)$$

$$\Rightarrow a^2\frac{(y_2-y_1)}{(x_2-x_1)}(y_2+y_1)=-b^2(x_2+x_1)$$

But $k=\frac{y_2-y_1}{x_2-x_1}$, so:

$$ka^2(y_2+y_1)=-b^2(x_2+x_1)$$

Using the coordinates of the midpoint of AB, $ka^2\times 2y=-b^2\times 2x$

So the locus of the midpoint of the chord is given by $ka^2y+b^2x=0$

This is a line passing through the origin.