Conic sections 2 3E

1 $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}-\frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x}=0$ which gives $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{b^2x}{a^2y}$

a $a^2=16,\ b^2=2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x}{8y}$

At (12, 4), $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{8}$

Equation of tangent is $y-4=\frac{3}{8}(x-12)$ or $8y=3x-4$

Equation of normal is $y-4=-\frac{8}{3}(x-12)$ or $3y+8x=108$

b $a^2=36,\ b^2=12 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x}{3y}$

At (12, 6), $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{3}$

Equation of tangent is $y-6=\frac{2}{3}(x-12)$ or $3y=2x-6$

Equation of normal is $y-6=-\frac{3}{2}(x-12)$ or $2y+3x=48$

c $a^2=25,\ b^2=3 \ \therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x}{25y}$ at (10, 3) $y'=\frac{2}{5}$

At (10, 3) equation of tangent is $y-3=\frac{2}{5}(x-10)$ or $5y=2x-5$

Equation of normal is $y-3=-\frac{5}{2}(x-10)$ or $2y+5x=56$

2 a $x=5\cosh t,\ y=2\sinh t \ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\cosh t}{5\sinh t}$

Equation of tangent is $y-2\sinh t=\frac{2\cosh t}{5\sinh t}(x-5\cosh t)$

or $5y\sinh t+10=2x\cosh t$

Equation of normal is $y-2\sinh t=-\frac{5\sinh t}{2\cosh t}(x-5\cosh t)$

or $2y\cosh t+5x\sinh t=29\cosh t\sinh t$

b $x=\sec t,\ y=3\tan t \ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3\sec^2 t}{\sec t\tan t}=\frac{3\sec t}{\tan t}$

Equation of tangent is $y-3\tan t=\frac{3\sec t}{\tan t}(x-\sec t)$ or $y\tan t+3=3x\sec t$

Equation of normal is $y-3\tan t=-\frac{\tan t}{3\sec t}(x-\sec t)$ or $3y\sec t+x\tan t=10\sec t\tan t$

3 Use the chain rule to find the gradient: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{b\sec^2 t}{a\tan t\sec t} = \dfrac{b\sec t}{a\tan t}$

Equation of tangent is $y - b\tan t = \dfrac{b\sec t}{a\tan t}(x - a\sec t)$

$$ay\tan t - ab\tan^2 t = bx\sec t - ab\sec^2 t$$
$$ay\tan t + ab(\sec^2 t - \tan^2 t) = bx\sec t$$

Recall that $\sec^2 t - \tan^2 t = 1$

Rearranging, $bx\sec t - ay\tan t = ab$

4 $x = a\cosh t,\ y = b\sinh t \Rightarrow \dfrac{dy}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{b\cosh t}{a\sinh t}$

Gradient of normal is $-\dfrac{a\sinh t}{b\cosh t}$

Equation of normal is $y - b\sinh t = -\dfrac{a\sinh t}{b\cosh t}(x - a\cosh t)$

$$by\cosh t - b^2\sinh t\cosh t = -ax\sinh t + a^2\cosh t\sinh t$$
$$ax\sinh t + by\cosh t = (a^2 + b^2)\sinh t\cosh t$$

5 $x = 4\cosh t,\ y = 3\sinh t \Rightarrow \dfrac{dy}{dx} = \dfrac{3\cosh t}{4\sinh t}$

Equation of tangent is $y - 3\sinh t = \dfrac{3\cosh t}{4\sinh t}(x - 4\cosh t)$

a At A, $x = 0 \Rightarrow y = 3\sinh t - \dfrac{3\cosh^2 t}{\sinh t} = -\dfrac{3}{\sinh t}$

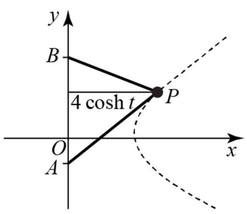
So is $\left(0, -\dfrac{3}{\sinh t}\right)$

b Using the result from question 4 with $a = 4,\ b = 3$

Equation of normal is $4x\sinh t + 3y\cosh t = (4^2 + 3^2)\sinh t\cosh t$
$$= 25\sinh t\cosh t$$

At B, $x = 0 \Rightarrow y = \dfrac{25}{3}\sinh t$ so B is $\left(0, \dfrac{25}{3}\sinh t\right)$

c



$$\text{Area of } \Delta APB = \frac{1}{2}\left|\left(\frac{25}{3}\sinh t - \left(-\frac{3}{\sinh t}\right)\right)4\cosh t\right|$$
$$= \frac{2}{3}\left|(25\sinh^2 t + 9)\coth t\right|$$

6 $\frac{x^2}{4}-\frac{y^2}{9}=1 \quad x=2\sec t,\ a=2$

$y=3\tan t,\ b=3$

From question 3 the equation of the tangent is:

$3x\sec t-2y\tan t=6$

Tangents meet at (1, 0), so let $x = 1$, $y = 0$

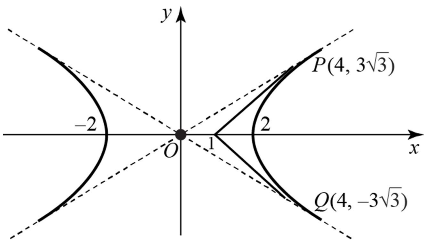
$\Rightarrow 3\sec t=6$

so $\quad \frac{1}{2}=\cos t$

Then $\quad t=\pm\frac{\pi}{3}$

$\sec\left(\pm\frac{\pi}{3}\right)=2, \quad \tan\left(\pm\frac{\pi}{3}\right)=\pm\sqrt{3}$

So the coordinates of P and Q are $(4,\ 3\sqrt{3})$ and $(4,\ -3\sqrt{3})$



7 Using the result $y=mx+c$ is a tangent to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for $b^2+c^2=a^2m^2$

$y=2x+c \quad \Rightarrow m=2$

$\frac{x^2}{10}-\frac{y^2}{4}=1 \Rightarrow a^2=10,\ b^2=4$

So $\quad 4+c^2=2^2\times 10=40$

$c^2=36$

$c=\pm 6$

8 Use the result $b^2+c^2=a^2m^2$ for $y=mx+c$ to be a tangent to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$y=mx+12 \Rightarrow c=12$

$\frac{x^2}{49}-\frac{y^2}{25}=1 \Rightarrow a^2=49,\ b^2=25$

So $\quad 25+12^2=49m^2$

$169=49m^2$

$m^2=\left(\frac{13}{7}\right)^2$

$m=\pm\frac{13}{7}$

9 Use the result $b^2+c^2=a^2m^2$ for $y=mx+c$ to be a tangent to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Using $\frac{x^2}{4}-\frac{y^2}{15}=1 \Rightarrow a^2=4,\ b^2=15$ so $15+c^2=4m^2$ **(1)**

Using $\frac{x^2}{9}-\frac{y^2}{95}=1 \Rightarrow a^2=9, b^2=95$ so $95+c^2=9m^2$ **(2)**

Solving the simultaneous equations:

(2) − (1) $80=5m^2$

$\Rightarrow$ $m^2=16$

$m=\pm 4$

Substituting $m=\pm 4$ into (1):

$c^2=4(16)-15$

$=49$

$\Rightarrow c=\pm 7$

So $m=\pm 4$ and $c=\pm 7$, i.e. lines $y=4x\pm 7$ and $y=-4x\pm 7$

10 a Use the result $b^2+c^2=a^2m^2$ for $y=mx+c$ to be a tangent to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$y=-x+c \Rightarrow m=-1$

Using $\frac{x^2}{25}-\frac{y^2}{16}=1 \Rightarrow a^2=25,\ b^2=16$

So $16+c^2=25(-1)^2$

$c^2=9$

$c=\pm 3$

But $c>0$, so $c=3$

b Substitute $y=(3-x)$ into the equation for the hyperbola

$$\frac{x^2}{25}-\frac{(3-x)^2}{16}=1$$

$16x^2-25(9+x^2-6x)=25\times 16$

$-9x^2-225+150x=400$

$0=9x^2-150x+625$

$0=(3x-25)^2$

$\Rightarrow x=\frac{25}{3},\ y=-\frac{16}{3}$

So P is $\left(\frac{25}{3}, \frac{-16}{3}\right)$

11 a $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$$x = a\cosh t,\ \ y = b\sinh t\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{b\cosh t}{a\sinh t}$$

Gradient of normal is $-\frac{a\sinh t}{b\cosh t}$

Equation of normal is $y-b\sinh t=-\frac{a\sinh t}{b\cosh t}(x-a\cosh t)$

$$by\cosh t-b^2\sinh t\cosh t=-ax\sinh t+a^2\cosh t\sinh t$$

$$ax\sinh t+by\cosh t=(a^2+b^2)\sinh t\cosh t$$

b At point P, $y=0$

Substituting $y=0$ in the equation for the normal: $ax=(a^2+b^2)\cosh t$

$$x=\frac{(a^2+b^2)}{a}\cosh t$$

The coordinates of P are $\left(\left(\frac{a^2+b^2}{a}\right)\cosh t,\ 0\right)$

c At the point $(a, 0)$, $y=b\sinh t=0$, which corresponds to $t=0$, since $b\neq 0$

Using the general form of the equation of the tangent to a hyperbola:

$$bx\cosh t-ay\sinh t=ab$$
$$bx=ab$$
$$x=a$$

So the equation of l_2 is $x=a$.

Substituting this into the equation of l_1 gives:

$$a^2\sinh t+by\cosh t=(a^2+b^2)\sinh t\cosh t$$
$$by\cosh t=a^2\sinh t(\cosh t-1)+b^2\sinh t\cosh t$$
$$y=\frac{a^2\sinh t(\cosh t-1)+b^2\sinh t\cosh t}{b\cosh t}$$

The coordinates of Q are then $\left(a,\frac{(a^2+b^2)\sinh t\cosh t-a^2\sinh t}{b\cosh t}\right)$

12 a Use the chain rule to find the gradient of the tangent to $\frac{x^2}{49}-\frac{y^2}{25}=1$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}}=\frac{5\sec^2\theta}{7\tan\theta\sec\theta}=\frac{5\sec\theta}{7\tan\theta}=\frac{5}{7\sin\theta}$$

An equation of the tangent is: $y-5\tan\theta=\frac{5}{7\sin\theta}(x-7\sec\theta)$

$$7y\sin\theta-35\tan\theta\sin\theta=5x-35\sec\theta$$
$$7y\sin\theta=5x-35\cos\theta$$

(It's easy to verify that the relation $\tan\theta\sin\theta-\sec\theta=-\cos\theta$ holds.)

12 b The gradient of a line that is perpendicular to l_1 is $-\dfrac{7\sin\theta}{5}$, therefore the equation of l_2 (which passes through the origin) is $y = -\dfrac{7\sin\theta}{5}x$

Substitute this value into the equation of l_1:

$$-\frac{49\sin^2\theta}{5}x = 5x - 35\cos\theta$$
$$-49x\sin^2\theta = 25x - 175\cos\theta$$
$$x = \frac{175\cos\theta}{25+49\sin^2\theta}$$

Then $y = -\dfrac{7\sin\theta}{5}x$

$$= -\frac{7\sin\theta}{5}\times\frac{175\cos\theta}{25+49\sin^2\theta}$$
$$= -\frac{245\sin\theta\cos\theta}{25+49\sin^2\theta}$$

The coordinates of Q are $\left(\dfrac{175\cos\theta}{25+49\sin^2\theta}, -\dfrac{245\sin\theta\cos\theta}{25+49\sin^2\theta}\right)$

13 $x^2 - 4y^2 = 16 \Rightarrow \dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$ so $a = 4$, $b = 2$

Let $P = (x_1, y_1)$, $Q = (x_2, y_2)$

Use the chain rule to find the gradient for a general point on the hyperbola $(4\cosh t, 2\sinh t)$:

gradient of the tangent is $\dfrac{\cosh t}{2\sinh t} = \dfrac{x}{4y}$

The equation of the tangent at P is then $y - y_1 = \dfrac{x}{4y}(x - x_1)$

$$4y^2 - 4yy_1 = x^2 - xx_1$$
$$xx_1 - 4yy_1 = 16$$

The same holds for the tangent at Q, so $xx_2 - 4yy_2 = 16$

The point (m, n) must satisfy both equations.

Then

$$mx_1 - 4ny_1 = mx_2 - 4ny_2 \quad \Rightarrow \quad m(x_1 - x_2) = 4n(y_1 - y_2)$$

Then the slope of the line l_1, which joins P and Q, is $\dfrac{m}{4n}$

But writing $y - y_1 = \dfrac{m}{4n}(x - x_1)$ gives $4ny - 4ny_1 = mx - mx_1$, and we already know that $mx_1 - 4ny_1 = 16$, so the equation of line l is $mx - 4ny = 16$

14 Consider the point $(4\sec\theta, 2\tan\theta)$. Differentiating using the chain rule (see question 3 in this exercise) leads to the equation for the tangent: $2x\sec\theta - 4y\tan\theta = 8$

Multiplying both sides by $\cos\theta$ gives $2x - 4y\sin\theta = 8\cos\theta$

Substitute $x = 6$ and $y = 4$ to get $4\sin\theta + 2\cos\theta = 3$

Let $R\sin\alpha = 4$ and $R\cos\alpha = 2$: this gives $R\cos(\theta - \alpha) = 3$

Find R from $R^2(\cos^2\alpha + \sin^2\alpha) = 4^2 + 2^2$ so $R = \sqrt{4^2 + 2^2} = \sqrt{20}$

Find α by calculating $\arctan\frac{4}{2} = \arctan 2 = 1.107\ldots$.

Using the condition $\sqrt{20}\cos(\theta - 1.107\ldots) = 3$ gives a set of values:

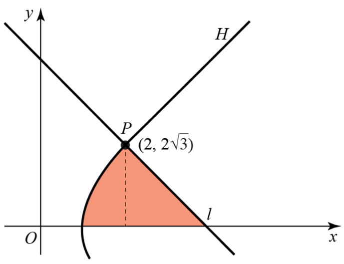
$\theta - 1.107\ldots = \ldots, 0.835\ldots, 5.447\ldots, 7.118\ldots, \ldots$

There are only two possible values for θ in the range $[0, 2\pi]$, so there are only two possible values for θ; therefore there are only two tangents.

15 Substituting $x = 2$ into the equation of the hyperbola gives the coordinates of P as $(2, 2\sqrt{3})$.

In the first quadrant the curve crosses the axis at (1, 0).

The area R is made up of two parts; the area under the hyperbola from $x = 1$ to $x = 2$, and the right-angled triangle from $x = 2$ to where line l crosses the x-axis.



Area under the hyperbola:

For $x>1$, in the first quadrant the hyperbola can be seen as the graph of the function $y = 2\sqrt{x^2 - 1}$, which can be integrated.

To solve the integral $\int_1^{2\sqrt{3}} 2\sqrt{x^2 - 1}\,dx$, use the substitution $x = \cosh u$:

$$2\int_0^{\text{arcosh}\,2} \sinh^2 u\,du = 2\int_0^{\text{arcosh}\,2} \frac{\cosh 2u - 1}{2}\,du$$

$$= -\text{arcosh}\,2 + \left[\frac{\sinh 2u}{2}\right]_0^{\text{arcosh}\,2}$$

$$= -\text{arcosh}\,2 + [\sinh(\text{arcosh}\,2)\cosh(\text{arcosh}\,2)]$$

$$= -\text{arcosh}\,2 + 2\sqrt{3}$$

Area of triangle:

Using implicit differentiation, gradient of the tangent is $\frac{dy}{dx} = \frac{4x}{y}$

so at P gradient of the tangent is $\frac{4\sqrt{3}}{3}$, and line l has gradient $-\frac{\sqrt{3}}{4}$

The normal at P has equation $y - 2\sqrt{3} = -\frac{\sqrt{3}}{4}(x - 2) \Rightarrow 4y + x\sqrt{3} = 10\sqrt{3}$

This meets the x-axis at $x = 10$. The area of the right-angled triangle contained in R is $\frac{8 \times 2\sqrt{3}}{2} = 8\sqrt{3}$

The total area of the region R is therefore $8\sqrt{3} - \text{arcosh}\,2 + 2\sqrt{3} = 10\sqrt{3} - \text{arcosh}\,2$

16 a The equations of the asymptotes of H are $y = x$ and $y = -x$.

Differentiating, gradient of tangent to H is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$

A and B lie on the lines $y = x$ and $y = -x$. Let A and B have coordinates (a, a) and $(b, -b)$.

The midpoint of AB is $\left(\frac{a+b}{2}, \frac{a-b}{2}\right)$

For a generic point P on H, the coordinates are (X, Y), so the gradient of the tangent at P is $\frac{X}{Y}$ and the equation of the tangent at P is $y - Y = \frac{X}{Y}(x - X)$

This tangent cuts the asymptotes at A and B, so the coordinates of A and B must be on the line.

At A: $a - Y = \frac{X}{Y}(a - X) \Rightarrow a = X + Y$

At B: $-b - Y = \frac{X}{Y}(b - X) \Rightarrow b = X - Y$

So $X = \frac{a+b}{2}$ and $Y = \frac{a-b}{2}$

b $|OA|$ is $\sqrt{2}|a|$ for all positions of A, and $|OB|$ is $\sqrt{2}|b|$ for all positions of B.

So $|OA| \times |OB| = 2|ab|$

From part **a**, $X^2 = \frac{a^2 + 2ab + b^2}{4}$ and $Y^2 = \frac{a^2 - 2ab + b^2}{4}$

So in terms of X and Y, $|ab| = |X^2 - Y^2|$ but from the equation for H this is equal to 1

So $|OA| \times |OB| = 2|ab| = 2$, which is constant.