Conic sections 2 3D

1 $x = a\cos\theta, y = b\sin\theta$

$\frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$

So tangent is

$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

Equation of tangent is $bx\cos\theta + ay\sin\theta = ab$

Normal gradient is $\frac{a\sin\theta}{b\cos\theta}$

So normal is $y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

Equation of normal is:

$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$

a $a = 2, b = 1$

So equation of tangent is:

$x\cos\theta + 2y\sin\theta = 2$

Equation of normal is:

$2x\sin\theta - y\cos\theta = 3\sin\theta\cos\theta$

b $\frac{x^2}{25} + \frac{y^2}{9} = 1 \Rightarrow a = 5, b = 3$

So equation of tangent is:

$3x\cos\theta + 5y\sin\theta = 15$

Equation of normal is:

$5x\sin\theta - 3y\cos\theta = 16\sin\theta\cos\theta$

2 a $\frac{x^2}{9} + y^2 = 1 \Rightarrow \frac{2x}{9} + 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{x}{9y}$ so at $\left(\sqrt{5}, \frac{2}{3}\right)$ $m = -\frac{\sqrt{5}}{6}$

Tangent at

$\left(\sqrt{5}, \frac{2}{3}\right)$ is $y - \frac{2}{3} = -\frac{\sqrt{5}}{6}(x - \sqrt{5})$

$\Rightarrow 6y + \sqrt{5}x = 9$

Normal at

$\left(\sqrt{5}, \frac{2}{3}\right)$ is $y - \frac{2}{3} = \frac{6}{\sqrt{5}}(x - \sqrt{5})$

$3\sqrt{5}y - 2\sqrt{5} = 18x - 18\sqrt{5}$

$\Rightarrow 3\sqrt{5}y = 18x - 16\sqrt{5}$

2 b $\frac{x^2}{16} + \frac{y^2}{4} = 1 \Rightarrow \frac{x}{8} + \frac{y}{2}\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{x}{4y}$ so at $(-2, \sqrt{3})$ $m = \frac{1}{2\sqrt{3}}$

Tangent at

$(-2, \sqrt{3})$ is $y - \sqrt{3} = \frac{1}{2\sqrt{3}}(x - (-2))$

$\Rightarrow 2\sqrt{3}y - x = 8$

Normal at

$(-2, \sqrt{3})$ is $y - \sqrt{3} = -2\sqrt{3}(x - (-2))$

$\Rightarrow y + 2\sqrt{3}x = -3\sqrt{3}$

3 Use the chain rule to find the gradient:

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b\cos t}{-a\sin t}$

Then the equation of the tangent is given by

$(y - b\sin t) = -\frac{b\cos t}{a\sin t}(x - a\cos t)$

$ay\sin t - ab\sin^2 t = -bx\cos t + ab\cos^2 t$

$bx\cos t + ay\sin t = ab$

4 a Using the method in Example 13, substitute for y in the equation of the ellipse.

$\frac{x^2}{4} + y^2 = 1 \Rightarrow \frac{x^2}{4} + (x + \sqrt{5})^2 = 1$

So $x^2 + 4(x^2 + 2\sqrt{5}x + 5) = 4$

$5x^2 + 8\sqrt{5}x + 16 = 0$

This has discriminant:

$\left(8\sqrt{5}\right)^2 - 4 \times 5 \times 16 = 0$

So the line meets the ellipse at only one point and therefore is a tangent to the ellipse.

4 b To find the point of contact, solve the equation from part **a** for x:

$5x^2+8\sqrt{5}x+16=0$

$(\sqrt{5}x+4)^2=0$

$\Rightarrow x=-\frac{4}{\sqrt{5}}=-\frac{4}{5}\sqrt{5}$

$\Rightarrow y=-\frac{4}{5}\sqrt{5}+\sqrt{5}=\frac{1}{5}\sqrt{5}$

So the point of contact is $\left(-\frac{4}{5}\sqrt{5},\frac{1}{5}\sqrt{5}\right)$

5 a $x=3\cos\theta, y=2\sin\theta\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\cos\theta}{-3\sin\theta}$

So gradient of normal is $\frac{3\sin\theta}{2\cos\theta}$

Equation of normal is:

$y-2\sin\theta=\frac{3\sin\theta}{2\cos\theta}(x-3\cos\theta)$

$2y\cos\theta-4\cos\theta\sin\theta$
$=3x\sin\theta-9\sin\theta\cos\theta$
$2y\cos\theta-3x\sin\theta$
$=-5\sin\theta\cos\theta$

5 b Normal at P crosses the x-axis at

$y=0, x=-\frac{5}{6}$

Substituting into the equation for the normal from part **a**:

$\frac{15}{6}\sin\theta=-5\sin\theta\cos\theta$

$\Rightarrow\sin\theta\left(\frac{1}{2}+\cos\theta\right)=0$

$\Rightarrow\sin\theta=0\text{ or }\cos\theta=-\frac{1}{2}$

$\sin\theta=0$ gives $\theta=0°$ or $180°$

$\cos\theta=-\frac{1}{2}$ gives $\theta=120°$ or $240°$

$\theta=0°\Rightarrow x=3, y=0$

$\theta=180°\Rightarrow x=-3, y=0$

$\theta=120°\Rightarrow x=\frac{3}{2}, y=\sqrt{3}$

$\theta=240°\Rightarrow x=-\frac{3}{2}, y=-\sqrt{3}$

So the coordinates of other possible positions of P are

$(3,0),(-3,0)\left(-\frac{3}{2},\sqrt{3}\right)$ or $\left(-\frac{3}{2},-\sqrt{3}\right)$

6 $y=mx+c$ is a tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

if $a^2m^2+b^2=c^2$

$y=2x+c\Rightarrow m=2, c=?$

$x^2+\frac{y^2}{4}=1\Rightarrow a=1, b=2$

$a^2m^2+b^2=c^2\Rightarrow 1\times4+4=c^2$

$c^2=8$ so $c=\pm2\sqrt{2}$

7 Substitute $y = mx + 3$ into the equation for the ellipse.

$$x^2 + \frac{(mx+3)^2}{5} = 1$$

$$5x^2 + (mx+3)^2 = 5$$

$$(5+m^2)x^2 + 6mx + 4 = 0$$

Since the line is a tangent the discriminant of this equation must equal zero (must have equal roots).

So $36m^2 = 4\times(5+m^2)\times 4 = 80 + 16m^2$

$$20m^2 = 80$$

$$m^2 = 4$$

$$\therefore \quad m = \pm 2$$

> The $a^2m^2 + b^2 = c^2$ condition could be used as in question 6.

8 a $y = mx + 4, \ \dfrac{x^2}{3} + \dfrac{y^2}{4} = 1$

$\Rightarrow c = 4, \ a^2 = 3, \ b^2 = 4$

Using the condition $a^2m^2 + b^2 = c^2$:

$$a^2m^2 + b^2 = c^2$$

$$\Rightarrow 3m^2 + 4 = 16$$

$$3m^2 = 12$$

$$m = \pm 2$$

But $m > 0$, so $m = 2$

b $y = 2x + 4, \dfrac{x^2}{3} + \dfrac{y^2}{4} = 1$

Substitute into the equation for the ellipse:

$$\frac{x^2}{3} + \frac{(4x^2+16x+16)}{4} = 1$$

$$\Rightarrow x^2 + 3x^2 + 12x + 12 = 3$$

$$4x^2 + 12x + 9 = 0$$

$$(2x+3)^2 = 0$$

$$x = -\frac{3}{2}, \ y = 2x + 4 = 1$$

So P is $(-\frac{3}{2}, 1)$

8 c Gradient of normal is $-\dfrac{1}{2}$

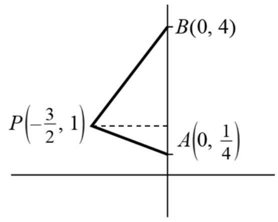
Equation of normal at P is

$$y - 1 = -\frac{1}{2}\left(x - \left(-\frac{3}{2}\right)\right)$$

$$x = 0 \Rightarrow y = 1 - \frac{3}{4} = \frac{1}{4}$$

So A is $(0, \frac{1}{4})$

d Tangent is $y = 2x + 4, \ x = 0 \Rightarrow y = 4$



So B is (0, 4)

Area of $\Delta APB = \dfrac{1}{2}\left(4 - \dfrac{1}{4}\right)\times\dfrac{3}{2}$

$$= \frac{1}{2}\times\frac{15}{4}\times\frac{3}{2} = \frac{45}{16}$$

9 a $\dfrac{dy}{d\theta} = 2\cos\theta, \ \dfrac{dx}{d\theta} = -3\sin\theta$

$$\frac{dy}{dx} = \frac{2\cos\theta}{-3\sin\theta} = -\frac{2}{3}\cot\theta$$

b $\dfrac{\left(\frac{9}{5}\right)^2}{9} + \dfrac{\left(\frac{-8}{5}\right)^2}{4} = \dfrac{9}{25} + \dfrac{16}{25} = 1 = \text{RHS}$

So $Q\left(\dfrac{9}{5}, -\dfrac{8}{5}\right)$ lies on E

c Let Q be the point $(3\cos\phi, 2\sin\phi)$

Using the coordinates of Q:

$$\frac{9}{5} = 3\cos\phi \Rightarrow \cos\phi = \frac{3}{5}$$

$$-\frac{8}{5} = 2\sin\phi \Rightarrow \sin\phi = -\frac{4}{5}$$

So $\cot\phi = -\dfrac{3}{4}$

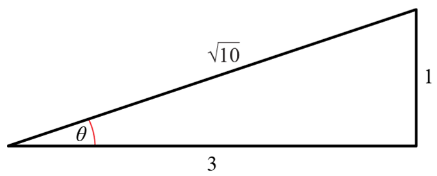
Gradient of tangent at Q is

$$-\frac{2}{3}\cot\phi = -\frac{2}{3}\times -\frac{3}{4} = \frac{1}{2}$$

9 d Tangent at P is perpendicular to tangent at Q, so gradient of tangent at $P = -2$

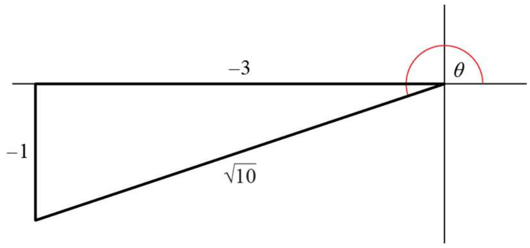
$$-2 = -\frac{2}{3}\cot\theta \Rightarrow \tan\theta = \frac{1}{3}$$

So P is $\left(\frac{9}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$



or

P is $\left(-\frac{9}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right)$



10 $y = mx + c$ and $\frac{x^2}{9} + \frac{y^2}{46} = 1$

$\Rightarrow a^2 = 9,\ b^2 = 46$

$\therefore b^2 + a^2m^2 = c^2 \Rightarrow 46 + 9m^2 = c^2$ **(1)**

$y = mx + c$ and $\frac{x^2}{25} + \frac{y^2}{14} = 1 \Rightarrow a^2 = 25,\ b^2 = 14$

$\therefore b^2 + a^2m^2 = c^2 \Rightarrow 14 + 25m^2 = c^2$ **(2)**

(1) − **(2)** $\Rightarrow 32 - 16m^2 = 0$

$\Rightarrow m^2 = 2$

$\therefore m = \pm\sqrt{2}$

$m^2 = 2$ and $14 + 25m^2 = c^2 \Rightarrow c^2 = 64$

$\therefore c = \pm 8$

So $m = \pm\sqrt{2},\ c = \pm 8$

11 Use the chain rule to find the gradient of the tangent:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\cos\theta}{-8\sin\theta} = -\frac{\cos\theta}{2\sin\theta}$$

The equation of the tangent l_1 is given by

$$y - 4\sin\theta = -\frac{\cos\theta}{2\sin\theta}(x - 8\cos\theta)$$

$$2y\sin\theta - 8\sin^2\theta = -x\cos\theta + 8\cos^2\theta$$

Equation of the tangent is
$x\cos\theta + 2y\sin\theta = 8$

Gradient of the normal l_2 is $\frac{2\sin\theta}{\cos\theta}$ and its equation is

$$y - 4\sin\theta = \frac{2\sin\theta}{\cos\theta}(x - 8\cos\theta)$$

$$y\cos\theta - 4\sin\theta\cos\theta = 2x\sin\theta - 16\sin\theta\cos\theta$$

Equation of the normal is:
$2x\sin\theta - y\cos\theta = 12\sin\theta\cos\theta$

Line l_1 cuts the x-axis at A, so $y = 0$:
$x\cos\theta = 8$ so $x = 8\sec\theta$
A is the point $(8\sec\theta, 0)$

Line l_2 cuts the y-axis at B, so $x = 0$:
$-y\cos\theta = 12\sin\theta\cos\theta$ so $y = -12\sin\theta$
B is the point $(0, -12\sin\theta)$

Now find the equation of the line AB.

$$\frac{y - 0}{x - 8\sec\theta} = \frac{0 - (-12\sin\theta)}{8\sec\theta - 0}$$

$$\frac{y}{12\sin\theta} = \frac{x - 8\sec\theta}{8\sec\theta}$$

$$2y\sec\theta = 3x\sin\theta - 24\sec\theta\sin\theta$$

$$3x\sin\theta - 2y\sec\theta = 24\sec\theta\sin\theta$$

$$3x\sin\theta\cos\theta - 2y = 24\sin\theta$$

12 a Use the chain rule to find the gradient:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos\theta}{-5\sin\theta}$$

Then the equation of l_1 is given by

$$y - 3\sin\theta = -\frac{3\cos\theta}{5\sin\theta}(x - 5\cos\theta)$$

$$5y\sin\theta - 15\sin^2\theta = -3x\cos\theta + 15\cos^2\theta$$

$$3x\cos\theta + 5y\sin\theta = 15$$

b At Q the line cuts the y-axis, so $x = 0$
Substitute in the equation for line l_1:

$$5y\sin\theta = 15 \text{ so } y = \frac{3}{\sin\theta}$$

The point Q has coordinates $\left(0, \frac{3}{\sin\theta}\right)$

The gradient of any line perpendicular to l_1 is:

$$\frac{5\sin\theta}{3\cos\theta}$$

Then the equation of l_2 is:

$$y - \frac{3}{\sin\theta} = \frac{5\sin\theta}{3\cos\theta}x$$

$$3y\sin\theta\cos\theta - 9\cos\theta = 5x\sin^2\theta$$

$$5x\sin^2\theta - 3y\sin\theta\cos\theta = -9\cos\theta$$

c If l_2 cuts the x-axis at $(-4, 0)$, then substituting into the equation for l_2 gives

$$-20\sin^2\theta = -9\cos\theta$$

$$20(1 - \cos^2\theta) = 9\cos\theta$$

$$20 - 20\cos^2\theta = 9\cos\theta$$

$$20\cos^2\theta + 9\cos\theta - 20 = 0$$

Using the quadratic formula to solve gives:

$$\cos\theta = \frac{-9 \pm \sqrt{81 - 4\times 20\times(-20)}}{40} = \frac{-9 \pm 41}{40}$$

Obviously $\cos\theta \neq -\frac{52}{40} < -1$, so

$$\cos\theta = \frac{32}{40} = \frac{4}{5}$$

13 a Use the chain rule to find the gradient:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\cos t}{-2\sin t} = -\frac{2\cos t}{\sin t}$$

Then an equation for l_1 is given by:

$$y - 4\sin t = -\frac{2\cos t}{\sin t}(x - 2\cos t)$$

$$y\sin t - 4\sin^2 t = -2x\cos t + 4\cos^2 t$$

$$2x\cos t + y\sin t = 4$$

b Since l_2 is perpendicular to l_1, the gradient of l_2 is $\frac{\sin t}{2\cos t}$

The line passes through the origin, so the equation is $y = mx \Rightarrow y = x\frac{\sin t}{2\cos t}$

Substituting $y = x\frac{\sin t}{2\cos t}$ into the equation of l_1 to find the coordinates of the intersection gives:

$$2x\cos t + x\frac{\sin^2 t}{2\cos t} = 4$$

$$\Rightarrow 4x\cos^2 t + x\sin^2 t = 8\cos t$$

$$\Rightarrow x = \frac{8\cos t}{4\cos^2 t + \sin^2 t}$$

Then $y = \frac{\sin t}{2\cos t} \times \frac{8\cos t}{4\cos^2 t + \sin^2 t} = \frac{4\sin t}{4\cos^2 t + \sin^2 t}$

The coordinates of Q are:

$$\left(\frac{8\cos t}{4\cos^2 t + \sin^2 t}, \frac{4\sin t}{4\cos^2 t + \sin^2 t}\right)$$

14 Use the chain rule to find the gradient:

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{b\cos t}{-a\sin t}$$

The equation of the tangent is:

$$y-b\sin t=\left(-\frac{b\cos t}{a\sin t}\right)(x-a\cos t)$$

$$ay\sin t-ab\sin^2 t=-bx\cos t+ab\cos^2 t$$

$$\Rightarrow bx\cos t+ay\sin t=ab$$

To find the x-intercept, substitute $y = 0$ in the equation for the tangent:

$$bx\cos t=ab\Rightarrow x=\frac{a}{\cos t}$$

To find the y-intercept, substitute $x = 0$ in the equation for the tangent:

$$ay\sin t=ab\Rightarrow y=\frac{b}{\sin t}$$

The area of the shaded triangle is:

$$\frac{1}{2}\times\frac{a}{\cos t}\times\frac{b}{\sin t}=\frac{ab}{2\sin t\cos t}$$

$$=\frac{ab}{\sin 2t}$$

$$=ab\operatorname{cosec}2t$$

15 Rearranging the equation for the ellipse:

$$y^2=4^2\left(1-\frac{x^2}{6^2}\right)\Rightarrow y=\frac{2\sqrt{36-x^2}}{3}$$

In the first Cartesian quadrant, the ellipse can be seen as the graph of the function

$$y=\frac{2\sqrt{36-x^2}}{3}$$

To find the area in the first quadrant, integrate from $x = 3$ to $x = 6$

The integral $\frac{2}{3}\int_3^6\sqrt{36-x^2}\,dx$ can be solved by substituting $6\sin u = x$, as follows:

$$\frac{2}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sqrt{36-36\sin^2 u}\,(6\cos u)\,du$$

$$=4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sqrt{1-\sin^2 u}\,(6\cos u)\,du$$

$$=24\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cos^2 u\,du$$

Using the identity $2\cos^2\theta\equiv 1+\cos 2\theta$

$$=12\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1+\cos 2u)\,du$$

$$=12\times\frac{\pi}{3}+\left[6\sin 2u\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$=4\pi+\left[-6\sin\frac{\pi}{3}\right]$$

$$=4\pi-3\sqrt{3}$$

The area of the shaded region is twice the value of the integral, so it is $8\pi-6\sqrt{3}$

Challenge

In the first Cartesian quadrant, the ellipse can be seen as the graph of the function $y = b\sqrt{1-\frac{x^2}{a^2}}$

This can be integrated. The integral $\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,\mathrm{d}x$ can be solved by substituting $x = a\sin u$, as follows:

$$\begin{aligned} b\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 u}\,(a\cos u)\,\mathrm{d}u &= ab\int_0^{\frac{\pi}{2}} \cos^2 u\,\mathrm{d}u \\ &= ab\int_0^{\frac{\pi}{2}} \frac{1+\cos 2u}{2}\,\mathrm{d}u \\ &= ab\left(\frac{\pi}{4}\right) + \frac{ab}{4}\left[\sin 2u\right]_0^{\frac{\pi}{2}} \\ &= ab\frac{\pi}{4} \end{aligned}$$

The area of the ellipse is four times the area contained in a single quadrant, so it is $ab\pi$