Conic sections 2 3C

1 a $a^2 = 9 \quad b^2 = 5$

$b^2 = a^2(1-e^2) \Rightarrow \frac{5}{9} = 1 - e^2$

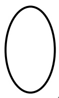
$\Rightarrow e^2 = \frac{4}{9}$ so $e = \frac{2}{3}$

b $a^2 = 16 \quad b^2 = 9$

$b^2 = a^2(1-e)^2 \Rightarrow \frac{9}{16} = 1 - e^2$

$\Rightarrow e^2 = \frac{7}{16}$ so $e = \frac{\sqrt{7}}{4}$

c $a^2 = 4 \quad b^2 = 8$

Need to use $a^2 = b^2(1-e^2)$ since $b > a$, so the ellipse is  shape.

$\frac{4}{8} = 1 - e^2 \Rightarrow e^2 = \frac{1}{2}$ so $e = \frac{1}{\sqrt{2}}$

2 a $a^2 = 4 \quad b^2 = 3$

$b^2 = a^2(1-e^2) \Rightarrow \frac{3}{4} = 1 - e^2$

$\Rightarrow e^2 = \frac{1}{4}$ so $e = \frac{1}{2}$

Foci are at $(\pm ae, 0) = (\pm 1, 0)$

Directrices are $x = \pm\frac{a}{e} \Rightarrow x = \pm 4$

b $a^2 = 16 \quad b^2 = 7$

$b^2 = a^2(1-e^2) \Rightarrow \frac{7}{16} = 1 - e^2$

$\Rightarrow e^2 = \frac{9}{16}$ so $e = \frac{3}{4}$

Foci are at $(\pm ae, 0) = (\pm 3, 0)$

Directrices are $x = \pm\frac{a}{e} \Rightarrow x = \pm\frac{16}{3}$

2 c $a^2 = 5, b^2 = 9$

Since $b > a$, use

$a^2 = b^2(1-e^2) \Rightarrow \frac{5}{9} = 1 - e^2$

$\Rightarrow e^2 = \frac{4}{9}$ so $e = \frac{2}{3}$

Foci are at $(0, \pm be)$, i.e. foci are $(0, \pm 2)$

Directrices are $y = \pm\frac{b}{e}$, i.e. $y = \pm\frac{9}{2}$

3 a Since the focus is on the x-axis and the directrix is parallel to the y-axis, we know that $a > b$. In fact, the major axis of the ellipse, on which the foci lie, has to be perpendicular to the directrix, so knowing that one focus is on the x-axis and that the directrix is perpendicular to it is enough to identify the major axis of the ellipse as lying on the x-axis.

b i The directrix is at $x = 12$, so $\frac{a}{e} = 12$

$\Rightarrow a = 12e$

The focus is at $(ae, 0)$, so $ae = 3$

$12e \times e = 3 \Rightarrow e^2 = \frac{1}{4}$ so $e = \frac{1}{2}$

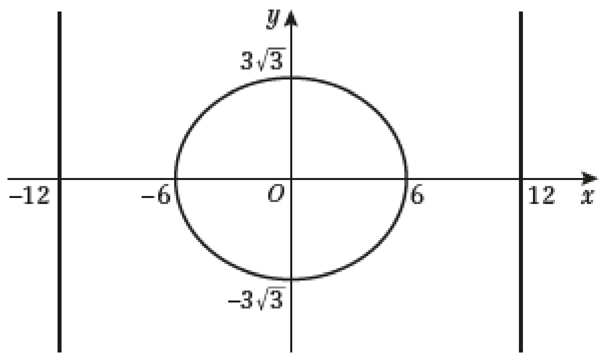
ii Since $ae = 3$, $a = 6$

Using $b^2 = a^2(1-e^2)$

$b^2 = 36\left(1 - \frac{1}{4}\right) = 36 \times \frac{3}{4} = 27$

$\Rightarrow b = 3\sqrt{3}$

c



4 a Since the directrix is parallel to the x-axis, and the focus is on the y-axis, we know that $b > a$.

b i The directrix is at $y = 8$, so $\frac{b}{e} = 8$

$\Rightarrow b = 8e$

The focus is at $(0, be)$, so $be = 2$

$8e \times e = 2 \Rightarrow e^2 = \frac{1}{4}$ so $e = \frac{1}{2}$

ii Since $be = 2$, $b = 4$

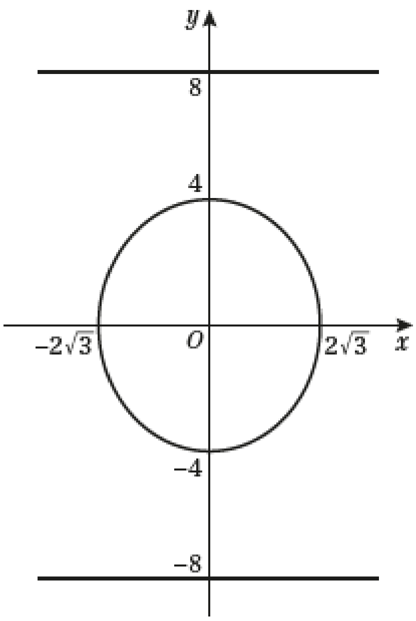
As $b > a$, use

$a^2 = b^2(1 - e^2)$

$a^2 = 16\left(1 - \frac{1}{4}\right) = 16 \times \frac{3}{4} = 12$

$\Rightarrow a = 2\sqrt{3}$

c



5 a $\frac{x^2}{5} - \frac{y^2}{3} = 1 \Rightarrow a^2 = 5, b^2 = 3$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{3}{5} = e^2 - 1$

$\Rightarrow e^2 = \frac{8}{5}$ so $e = \frac{2\sqrt{2}}{\sqrt{5}} = \frac{2\sqrt{10}}{5}$

b $\frac{x^2}{9} - \frac{y^2}{7} = 1 \Rightarrow a^2 = 9, b^2 = 7$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{7}{9} = e^2 - 1$

$\Rightarrow e^2 = \frac{16}{9}$ so $e = \frac{4}{3}$

5 c $\frac{x^2}{9} - \frac{y^2}{16} = 1 \Rightarrow a^2 = 9, b^2 = 16$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{16}{9} = e^2 - 1$

$\Rightarrow e^2 = \frac{25}{9}$ so $e = \frac{5}{3}$

6 a $\frac{x^2}{4} - \frac{y^2}{8} = 1$

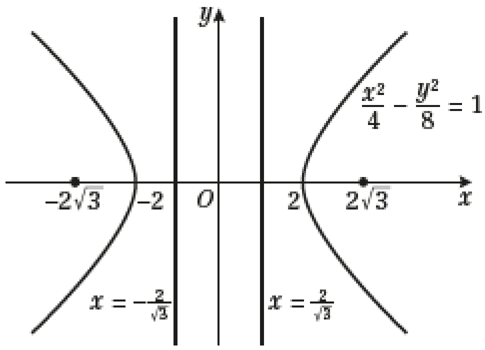
$a = 2, b = 2\sqrt{2}$

$b^2 = a^2(e^2 - 1) \Rightarrow \frac{8}{4} = e^2 - 1$

$\Rightarrow e = \sqrt{3}$

Foci are at $(\pm 2\sqrt{3}, 0)$

Directrices are $x = \pm\frac{2}{\sqrt{3}}$



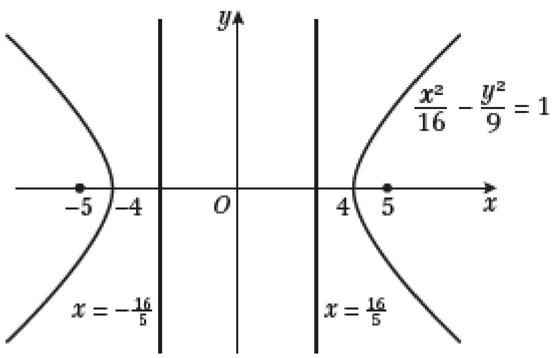
b $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$a = 4, b = 3$

$\Rightarrow 9 = 16(e^2 - 1) \Rightarrow e^2 = \frac{25}{16} \Rightarrow e = \frac{5}{4}$

Foci are at $(\pm 5, 0)$

Directrices are $x = \pm\frac{16}{5}$



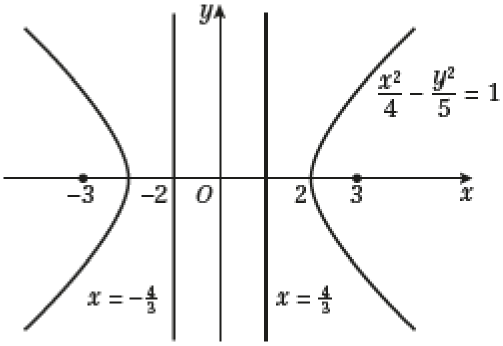
6 c $\frac{x^2}{4} - \frac{y^2}{5} = 1$

$a = 2,\ b = \sqrt{5}$

$\Rightarrow 5 = 4(e^2 - 1) \Rightarrow e^2 = \frac{9}{4} \Rightarrow e = \frac{3}{2}$

Foci are at (±3, 0)

Directrices are $x = \pm\frac{4}{3}$



7 a The eccentricity, e, of a hyperbola is given by $b^2 = a^2(e^2 - 1)$

Rearranging, $e = \sqrt{1 + \frac{b^2}{a^2}}$

and the foci have coordinates $(\pm ae, 0)$

Then we just need to compute the eccentricity in each case, as follows:

i $e = \sqrt{1 + \frac{1}{24}} = \frac{5}{\sqrt{24}}$;

then $ae = \frac{5}{\sqrt{24}} \times \sqrt{24} = 5$

ii $e = \sqrt{1 + 24} = 5$;

then $ae = 5 \times 1 = 5$

iii $e = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$;

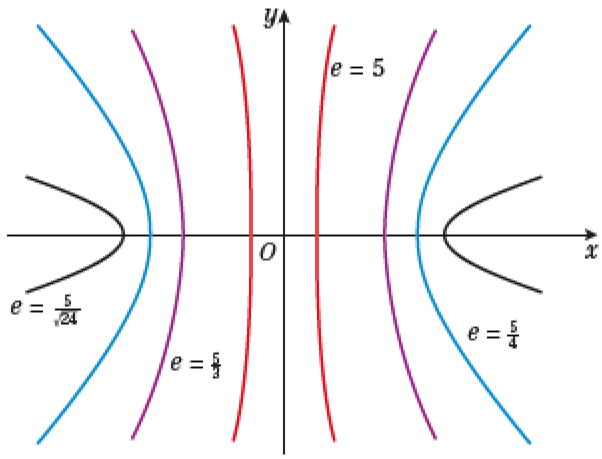
then $ae = \frac{5}{4} \times 4 = 5$

iv $e = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$;

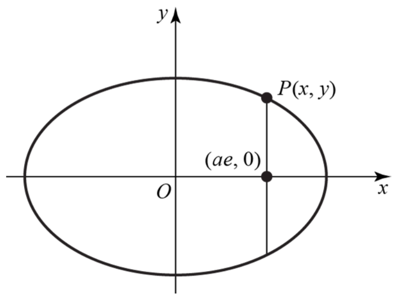
then $ae = \frac{5}{3} \times 3 = 5$

Since for all four hyperbolas $ae = 5$, all have foci at (±5, 0)

b We have already found the values of the eccentricity: $\frac{5}{\sqrt{24}}$, 5, $\frac{5}{4}$ and $\frac{5}{3}$



8 Since $a > b$, the ellipse has its major axis along the x-axis. Let P be a point of intersection of the chord with the ellipse, with coordinates (x, y).



The focus $(ae, 0)$ is on the chord, so $x = ae$. Substitute into the equation for the ellipse:

$\frac{(ae)^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow y^2 = b^2(1 - e^2)$

From the definition of eccentricity,

$b^2 = a^2(1 - e^2)$ so $e^2 = 1 - \frac{b^2}{a^2}$

Substituting for e^2 in the equation for y,

$y^2 = b^2\left(1 - 1 + \frac{b^2}{a^2}\right) \Rightarrow y = \pm\frac{b^2}{a}$

The length of the latus rectum is $2y = \frac{2b^2}{a}$

9 a Assume the foci are on the x-axis. The distance between the foci is $2ae = 16$, so $ae = 8$

The distance between the directrices is

$$\frac{2a}{e} = 25$$

Substituting for a gives

$$25e^2 = 16 \Rightarrow e = \frac{4}{5}$$

b The foci are on the y-axis, thus

$$b = \frac{8}{e} = 8 \times \frac{5}{4} = 10$$

$$\text{and } a = \left(\sqrt{1 - \frac{16}{25}}\right)10 = \frac{3}{5} \times 10 = 6$$

Then the equation of the ellipse is

$$\frac{x^2}{36} + \frac{y^2}{100} = 1$$

10 Rewrite the equation of the ellipse by dividing both sides by 36

$$\frac{x^2}{36} + \frac{y^2}{9} = 1 \text{ so } a = 6 \text{ and } b = 3$$

Then the eccentricity of the ellipse is

$$e = \sqrt{1 - \frac{9}{36}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

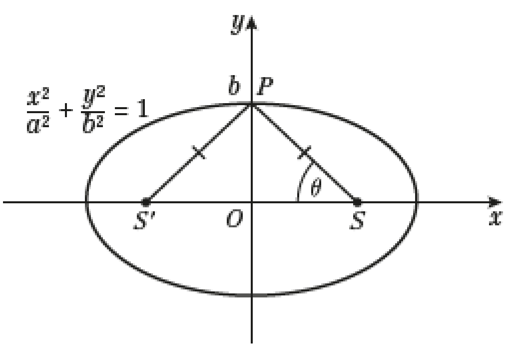
The points A and B have coordinates $(\pm ae, 0)$, so they are the foci of the ellipse.

Using the focus and directrix definitions of an ellipse, for any point P with coordinates (x, y),

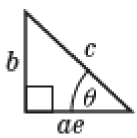
$$PA + PB = e\left(\frac{a}{e} + x\right) + e\left(\frac{a}{e} - x\right)$$

$$= 2a = 12$$

11



Consider ΔPOS



$$c^2 = b^2 + a^2e^2, \text{ but } b^2 = a^2(1 - e^2)$$

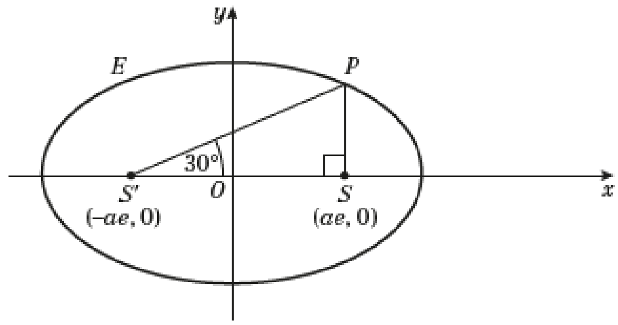
$$\Rightarrow c^2 = a^2 - a^2e^2 + a^2e^2 = a^2$$

$$\Rightarrow c = a$$

$$\text{So } \cos\theta = \frac{ae}{a} = e$$

> If you use the result that $SP + S'P = 2a$ then since $S'P = SP$ it is clear $SP = a$
>
> Hence $\cos\theta = \frac{ae}{a} = e$

12



$$PS \text{ is } y \text{ where } \frac{a^2e^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y^2 = b^2(1 - e^2)$$

$$y = b\sqrt{1 - e^2}$$

$$SS' = 2ae$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{y}{2ae} = \frac{b\sqrt{1 - e^2}}{2ae}$$

$$\text{But } b^2 = a^2(1 - e^2)$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{a\sqrt{1 - e^2}\sqrt{1 - e^2}}{2ae}$$

$$\Rightarrow \frac{2e}{\sqrt{3}} = 1 - e^2$$

$$\Rightarrow e^2 + \frac{2}{\sqrt{3}}e - 1 = 0$$

Completing the square,

$$e^2 + \frac{2}{\sqrt{3}}e + \frac{1}{3} = 1 + \frac{1}{3}$$

$$\Rightarrow \left(e + \frac{1}{\sqrt{3}}\right)^2 = 1 + \frac{1}{3} = \frac{4}{3}$$

$$\Rightarrow e + \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}} \text{ so } e = \frac{1}{\sqrt{3}} \quad (e > 0)$$