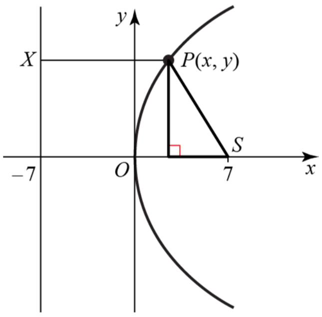
Conic Sections 1 2G

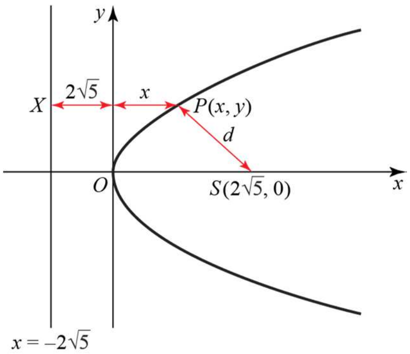
1



$$PS^2 = PX^2$$
$$(7-x)^2 + y^2 = (x+7)^2$$
$$x^2 - 14x + 49 + y^2 = x^2 + 14x + 49$$
$$y^2 = 28x$$

Since $y^2 = 4ax$, it follows that $a = 7$

2



From sketch the locus satisfies $SP = XP$.
Therefore, $SP^2 = XP^2$.

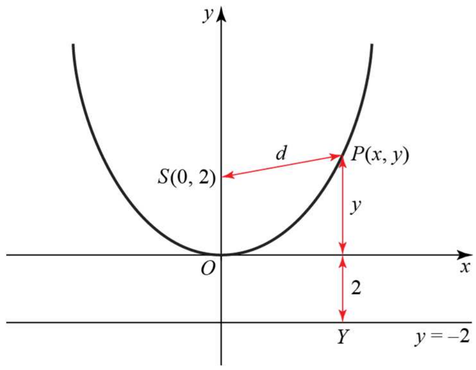
So, $(x-2\sqrt{5})^2 + (y-0)^2 = (x - -2\sqrt{5})^2$

$$x^2 - 4\sqrt{5}x + 20 + y^2 = x^2 + 4\sqrt{5}x + 20$$
$$-4\sqrt{5}x + y^2 = 4\sqrt{5}x$$

which simplifies to $y^2 = 8\sqrt{5}x$

So, the locus of P has an equation of the form $y^2 = 4ax$, where $a = 2\sqrt{5}$

3 a



From sketch the locus satisfies $SP = YP$.

Therefore, $SP^2 = YP^2$.

So, $(x-0)^2 + (y-2)^2 = (y - -2)^2$

$$x^2 + y^2 - 4y + 4 = y^2 + 4y + 4$$
$$x^2 - 4y = 4y$$

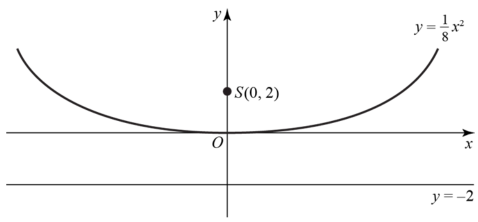
which simplifies to $x^2 = 8y$ and then

$$y = \frac{1}{8}x^2$$

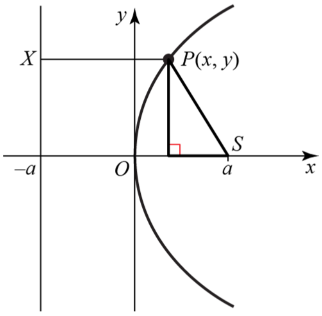
So, the locus of P has an equation of the form $y = \frac{1}{8}x^2$, where $k = \frac{1}{8}$

b The focus and directrix of a parabola with equation
$y^2 = 4ax$ are $(a, 0)$ and $x + a = 0$
respectively.
Therefore it follows that the focus and directrix of a parabola with equation
$x^2 = 4ay$ are $(0, a)$
and $y + a = 0$ respectively.
So the focus has coordinates (0, 2) and the directrix has equation $y + 2 = 0$

c



4



$$PS^2 = PX^2$$
$$(a-x)^2 + y^2 = (x+a)^2$$
$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$
So $y^2 = 4ax$

5 a Let X be the point on the line $x+3=0$ such that XP is horizontal.
Then $PS = PX$, so $PS^2 = PX^2$
$$(x-3)^2 + y^2 = (x+3)^2$$
$$x^2 - 6x + 9 + y^2 = x^2 + 6x + 9$$
$$y^2 = 12x$$
So $k = 12$

b Substitute the point $Q\left(x, 6\sqrt{6}\right)$ into the equation $y^2 = 12x$
$$\left(6\sqrt{6}\right)^2 = 12x$$
$$216 = 12x$$
$$x = 18$$

The equation of SQ is therefore
$$\frac{y-0}{x-3} = \frac{6\sqrt{6}-0}{18-3} = \frac{6\sqrt{6}}{15} = \frac{2\sqrt{6}}{5}$$
$$y = \frac{2\sqrt{6}}{5}(x-3)$$
$$y = \frac{2\sqrt{6}}{5}x - \frac{6\sqrt{6}}{5}$$

5 c Solving $y = \frac{2\sqrt{6}}{5}x - \frac{6\sqrt{6}}{5}$ and $y^2 = 12x$ simultaneously:
$$\left(\frac{2\sqrt{6}}{5}x - \frac{6\sqrt{6}}{5}\right)^2 = 12x$$
$$\frac{24x^2}{25} - \frac{144x}{25} + \frac{216}{25} = 12x$$
$$\frac{2x^2}{25} - \frac{12x}{25} + \frac{18}{25} = x$$
$$2x^2 - 12x + 18 = 25x$$
$$2x^2 - 37x + 18 = 0$$
$$(2x-1)(x-18) = 0$$
$x \neq 18$ (as $x = 18$ at Q),
so $2x - 1 = 0$ and so $x = \frac{1}{2}$

When $x = \frac{1}{2}$, $y^2 = 12 \times \left(\frac{1}{2}\right) = 6$,
so $y = -\sqrt{6}$

The coordinate of R is therefore $R\left(\frac{1}{2}, -\sqrt{6}\right)$

d Area of the trapezium
$$QRVW = \frac{1}{2} \times (WQ + VR) \times VW$$
$$= \frac{1}{2} \times \left(21 + \frac{7}{2}\right) \times 7\sqrt{6}$$
$$= \frac{1}{2} \times \frac{49}{2} \times 7\sqrt{6}$$
$$= \frac{343\sqrt{6}}{4}$$

6 The rectangular hyperbola $xy = c^2$ has the general point $P\left(ct, \frac{c}{t}\right)$
The coordinate of Q is therefore
$$Q(X, Y) = Q\left(ct, \frac{c}{2t}\right)$$
So $XY = ct\left(\frac{c}{2t}\right) = \frac{c^2}{2} = \left(\frac{c}{\sqrt{2}}\right)^2$

Therefore $k = \frac{c}{\sqrt{2}}$

7 a, b

Let $A(a, 0)$ and $B(0, b)$ be the points on the coordinate axes.

The area of the triangle AOB is $\frac{ab}{2} = q$, where q is a constant.

The midpoint of AB has coordinates $\left(\frac{a}{2}, \frac{b}{2}\right)$

So the coordinates (x, y) of M will always satisfy $xy = \frac{a}{2} \times \frac{b}{2} = \frac{ab}{4} = \frac{q}{2}$

This is of the form $xy = c^2$, where $c^2 = \frac{q}{2}$

Challenge

Each crease-line l is a perpendicular bisector between the point $S(a,0)$ and a point T on the line $x + a = 0$

Let a general point T on the line $x + a = 0$ have coordinate $(-a, Y)$

The gradient of ST is $\frac{Y - 0}{-a - a} = -\frac{Y}{2a}$

The gradient of the perpendicular bisector of ST is therefore $\frac{2a}{Y}$

The midpoint M is ST is

$$M\left(\frac{-a + a}{2}, \frac{Y + 0}{2}\right) = M\left(0, \frac{Y}{2}\right)$$

The equation of the perpendicular bisector, l (i.e. the crease line), of ST is therefore $y = \frac{2ax}{Y} + \frac{Y}{2}$

To find the point of intersection of l and $y^2 = 4ax$,

solve $y = \frac{2ax}{Y} + \frac{Y}{2}$ and $y^2 = 4ax$

simultaneously:

$$\left(\frac{2ax}{Y} + \frac{Y}{2}\right)^2 = 4ax$$

$$\frac{4a^2x^2}{Y^2} + \frac{Y^2}{4} + 2ax = 4ax$$

$$16a^2x^2 + Y^4 + 8axY^2 = 16axY^2$$

$$16a^2x^2 - 8axY^2 + Y^4 = 0$$

The discriminant '$b^2 - 4ac$'

$= (-8aY^2)^2 - 4 \times (16a^2) \times Y^4$

$= 64a^2Y^4 - 64a^2Y^4 = 0$

Therefore the crease line l only touches the parabola once.

Therefore the crease line l is a tangent to the parabola $y^2 = 4ax$