Conic Sections 1 2F

1 a $y^2 = 12x$

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 12$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{y}$

At $P(3t^2, 6t)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6}{6t} = \frac{1}{t}$

Tangent is:

$$y - 6t = \frac{1}{t}(x - 3t^2)$$
$$ty - 6t^2 = x - 3t^2$$
$$yt = x - 3t^2 + 6t^2$$
$$yt = x + 3t^2$$

The equation of the tangent to C at P is $yt = x + 3t^2$

b Gradient of tangent at $P(3t^2, 6t)$ is $m_T = \frac{1}{t}$

So gradient of normal at $P(3t^2, 6t)$ is

$$m_N = \frac{-1}{\left(\frac{1}{t}\right)} = -t$$

Normal is:

$$y - 6t = -t(x - 3t^2)$$
$$y - 6t = -tx + 3t^3$$
$$xt + y = 3t^3 + 6t$$

The equation of the normal to C at P is $xt + y = 3t^3 + 6t$

2 a $H: xy = 36 \Rightarrow y = 36x^{-1}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -36x^{-2} = -\frac{36}{x^2}$

At $P\left(6t, \frac{6}{t}\right)$,

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{36}{(6t)^2} = -\frac{36}{36t^2} = -\frac{1}{t^2}$

Tangent is:

$$y - \frac{6}{t} = -\frac{1}{t^2}(x - 6t)$$
$$t^2y - 6t = -(x - 6t)$$
$$t^2y - 6t = -x + 6t$$
$$x + t^2y = 6t + 6t$$
$$x + t^2y = 12t$$

The equation of the tangent to H at P is $x + t^2y = 12t$

b Gradient of tangent at $P\left(6t, \frac{6}{t}\right)$ is

$$m_T = -\frac{1}{t^2}$$

So gradient of normal at $P\left(6t, \frac{6}{t}\right)$ is

$$m_N = \frac{-1}{\left(-\frac{1}{t^2}\right)} = t^2$$

Normal is:

$$y - \frac{6}{t} = t^2(x - 6t)$$
$$ty - 6 = t^3(x - 6t)$$
$$ty - 6 = t^3x - 6t^4$$
$$6t^4 - 6 = t^3x - ty$$
$$6(t^4 - 1) = t^3x - ty$$

The equation of the normal to H at P is $t^3x - ty = 6(t^4 - 1)$

3 a Substituting $x = 5t^2$ and $y = 10t$ into $y^2 = 4ax$ gives

$$(10t)^2 = 4a(5t^2)$$

$$100t^2 = 20t^2a$$

$$a = \frac{100t^2}{20t^2} = 5$$

So, $a = 5$

b When $a = 5, y^2 = 4(5)x \Rightarrow y^2 = 20x$

$$2y\frac{dy}{dx} = 20 \text{ so } \frac{dy}{dx} = \frac{10}{y}$$

At $P(5t^2, 10t)$, $\frac{dy}{dx} = \frac{10}{10t} = \frac{1}{t}$

Tangent is:

$$y - 10t = \frac{1}{t}(x - 5t^2)$$

$$ty - 10t^2 = x - 5t^2$$

$$yt = x - 5t^2 + 10t^2$$

$$yt = x + 5t^2$$

Therefore, the equation of the tangent to C at P is $yt = x + 5t^2$

3 c Tangent is $yt = x + 5t^2$

Tangent cuts x-axis at X

$y = 0 \Rightarrow 0 = x + 5t^2$

$\Rightarrow x = -5t^2$

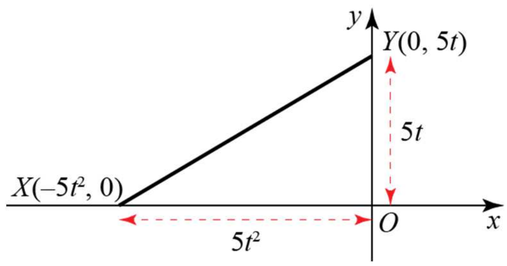
$\therefore$ Coordinates of X are $(-5t^2, 0)$

Tangent cuts y-axis at Y

$x = 0 \Rightarrow yt = 5t^2$

$\Rightarrow y = 5t$

$\therefore$ Coordinates of Y are $(0, 5t)$



Using sketch drawn,

$$\text{Area } \Delta OXY = \frac{1}{2}(5t^2)(5t)$$

$$= \frac{25}{2}t^3$$

Therefore, Area $\Delta OXY = \frac{25}{2}t^3$

4 a $y^2 = 4ax$

$$2y\frac{dy}{dx} = 4a \text{ so } \frac{dy}{dx} = \frac{2a}{y}$$

At $P(at^2, 2at)$, $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$

Tangent is:

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$ty - 2at^2 = x - at^2$$

$$ty = x - at^2 + 2at^2$$

$$ty = x + at^2$$

The equation of the tangent to C at P is $ty = x + at^2$

4 b The equation of a general tangent to C at any point P is $ty = x + at^2$

The tangents to C at A and B both pass through $(-4a, 3a)$, so substitute $x = -4a$ and $y = 3a$ into $ty = x + at^2$

$$t(3a) = -4a + at^2$$
$$0 = at^2 - 3at - 4a$$
$$t^2 - 3t - 4 = 0$$
$$(t+1)(t-4) = 0$$
$$t = -1 \text{ or } 4$$

When $t = -1$,
$x = a(-1)^2 = a,\ y = 2a(-1) = -2a$
$\Rightarrow (a, -2a)$

When $t = 4$,
$x = a(4)^2 = 16a, y = 2a(4) = 8a$
$\Rightarrow (16a, 8a)$

The coordinates of A and B are $(a, -2a)$ and $(16a, 8a)$

5 a $H: xy = 16 \Rightarrow y = 16x^{-1}$
$$\frac{dy}{dx} = -16x^{-2} = -\frac{16}{x^2}$$

At $P\left(4t, \frac{4}{t}\right)$,
$$\frac{dy}{dx} = -\frac{16}{(4t)^2} = -\frac{16}{16t^2} = -\frac{1}{t^2}$$

Tangent is:
$$y - \frac{4}{t} = -\frac{1}{t^2}(x - 4t)$$
$$t^2 y - 4t = -(x - 4t)$$
$$t^2 y - 4t = -x + 4t$$
$$x + t^2 y = 4t + 4t$$
$$x + t^2 y = 8t$$

The equation of the tangent to H at P is $x + t^2 y = 8t$

b A general parabola with equation $y^2 = 4ax$ has directrix $x + a = 0$
Here $y^2 = 16x \Rightarrow 4a = 16$,
giving $a = \frac{16}{4} = 4$
So the directrix has equation $x + 4 = 0$ or $x = -4$

Therefore at X, $x = -4$ and, as stated, $y = 5$
The coordinates of X are $(-4, 5)$

c The equation of a general tangent to H at any point P is $x + t^2 y = 8t$

The tangents to H at A and B both pass through $(-4, 5)$, so substitute $x = -4$ and $y = 5$ into $x + t^2 y = 8t$

$$(-4) + t^2(5) = 8t$$
$$5t^2 - 4 = 8t$$
$$5t^2 - 8t - 4 = 0$$
$$(t-2)(5t+2) = 0$$
$$t = 2 \text{ or } -\frac{2}{5}$$

When $t = 2$,
$$x = 4(2) = 8, y = \frac{4}{2} = 2$$
$\Rightarrow (8, 2)$

When $t = -\frac{2}{5}$,
$$x = 4\left(-\frac{2}{5}\right) = -\frac{8}{5}, y = \frac{4}{\left(-\frac{2}{5}\right)} = -10$$
$$\Rightarrow \left(-\frac{8}{5}, -10\right)$$

The coordinates of A and B are $(8, 2)$ and $\left(-\frac{8}{5}, -10\right)$

5 d Substitute $t=2$ and $t=-\frac{2}{5}$ into $x+t^2y=8t$ to find the equations of the tangents to H that go through the point X.

When $t=2$, tangent is
$x+4y=16 \Rightarrow x+4y-16=0$

When $t=-\frac{2}{5}$, tangent is

$$x+\left(-\frac{2}{5}\right)^2 y=8\left(-\frac{2}{5}\right)$$
$$x+\frac{4}{25}y=-\frac{16}{5}$$
$$25x+4y=-80$$
$$25x+4y+80=0$$

6 a $y^2=4ax$

$2y\frac{\mathrm{d}y}{\mathrm{d}x}=4a$ so $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2a}{y}$

At $P(at^2,2at)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2a}{2at}=\frac{1}{t}$

Tangent is:

$$y-2at=\frac{1}{t}(x-at^2)$$
$$ty-2at^2=x-at^2$$
$$ty=x-at^2+2at^2$$
$$ty=x+at^2$$

The equation of the tangent to C at P is
$ty=x+at^2$

This tangent cuts x-axis at A, so
y-coordinate of A is $y=0$

At A, $0=x+at^2 \Rightarrow x=-at^2$

The coordinates of A are $P(-at^2,0)$

6 b Gradient of tangent at
$P(at^2,2at)$ is $m_T=\frac{1}{t}$

So gradient of normal at
$P(at^2,2at)$ is $m_N=\frac{-1}{\left(\frac{1}{t}\right)}=-t$

Normal is:

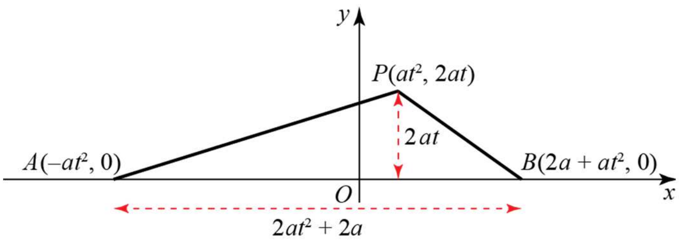
$$y-2at=-t(x-at^2)$$
$$y-2at=-tx+at^3$$

Normal cuts x-axis $\Rightarrow y=0$
So,

$$0-2at=-tx+at^3$$
$$tx=2at+at^3$$
$$x=2a+at^2$$

The coordinates of B are $(2a+at^2,0)$

6 c



Using sketch drawn,

$$\text{Area } \Delta APB = \frac{1}{2}(2a + 2at^2)(2at)$$
$$= at(2a + 2at^2)$$
$$= 2a^2t(1 + t^2)$$

Therefore, Area $\Delta APB = 2a^2t(1 + t^2)$

7 a $y^2 = 8x$

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 8$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{y}$

At $P(2t^2, 4t)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4}{4t} = \frac{1}{t}$

Gradient of tangent to C at $P(2t^2, 4t)$ is

$m_T = \frac{1}{t}$

So gradient of normal to C at

$P(t^2, 4t)$ is $m_N = \frac{-1}{\left(\frac{1}{t}\right)} = -t$

Normal is:

$y - 4t = -t(x - 2t^2)$

$y - 4t = -tx + 2t^3$

$xt + y = 2t^3 + 4t$

The equation of the normal to C at P

$xt + y = 2t^3 + 4t$

7 b The equation of a general normal to C at any point P is $xt + y = 2t^3 + 4t$

The normals to C at R, S and T all pass through (12, 0)

So substitute $x = -12$ and $y = 0$ into $xt + y = 2t^3 + 4t$

$$(12)t + 0 = 2t^3 + 4t$$
$$12t = 2t^3 + 4t$$
$$0 = 2t^3 + 4t - 12t$$
$$0 = 2t^3 - 8t$$
$$t^3 - 4t = 0$$
$$t(t^2 - 4) = 0$$
$$t(t-2)(t+2) = 0$$
$$t = 0, 2, -2$$

When $t = 0$,

$x = 2(0^2) = 0,\quad y = 4(0) = 0$

$\Rightarrow (0,\ 0)$

When $t = 2$,

$x = 2(2^2) = 8,\quad y = 4(2) = 8$

$\Rightarrow (8,\ 8)$

When $t = -2$,

$x = 2(-2)^2 = 8,\quad y = 4(-2) = -8$

$\Rightarrow (8,\ -8)$

The coordinates of R, S and T are (0, 0), (8, 8) and (8, –8)

7 c Substitute $t = 0, 2, -2$ into $xt + y = 2t^3 + 4t$ to find the equations of the normals N_1, N_2 and N_3 to C that go through the point (12, 0)

$N_1 : t = 0 \Rightarrow 0 + y = 0 + 0 \Rightarrow y = 0$

$$N_2 : t = 2 \Rightarrow x(2) + y = 2(8) + 4(2)$$
$$2x + y = 24$$
$$2x + y - 24 = 0$$

$$N_3 : t = -2 \Rightarrow x(-2) + y = 2(-8) + 4(-2)$$
$$-2x + y = -24$$
$$2x - y - 24 = 0$$

Hence the equations of the normals are $y = 0,\ 2x + y - 24 = 0$ and $2x - y - 24 = 0$

8 a Tangent to C at P has equation $ty = x + at^2$
(See question **6a** if you need help to understand this part).

tangent meets y-axis at $Q \Rightarrow x = 0$ at Q
Sub $x = 0$ into equation of tangent:

$$ty = 0 + at^2 \Rightarrow y = \frac{at^2}{t} \Rightarrow y = at$$

The coordinates of Q are $(0, at)$

b The focus of a parabola with equation $y^2 = 4ax$ has coordinates $(a, 0)$

So, the coordinates of S are $(a, 0)$

c $P(at^2, 2at)$, $Q(0, at)$ and $S(a, 0)$

$$m_{PQ} = \frac{at - 2at}{0 - at^2} = \frac{-at}{-at^2} = \frac{1}{t}$$
$$m_{SQ} = \frac{0 - at}{a - 0} = -\frac{at}{a} = -t$$

Therefore, $m_{PQ} \times m_{SQ} = \frac{1}{t} \times -t = -1$
So PQ is perpendicular to SQ

9 a $y^2 = 24x$

$$2y\frac{dy}{dx} = 24 \text{ so } \frac{dy}{dx} = \frac{12}{y}$$

At $P(6t^2, 12t)$, $\frac{dy}{dx} = \frac{12}{12t} = \frac{1}{t}$

Tangent is:

$$y - 12t = \frac{1}{t}(x - 6t^2)$$
$$ty - 12t^2 = x - 6t^2$$
$$ty = x - 6t^2 + 12t^2$$
$$ty = x + 6t^2$$

b The directrix of a parabola with equation $y^2 = 4ax$ has equation $x + a = 0$

Here, $y^2 = 24x \Rightarrow 4a = 24 \Rightarrow a = \frac{24}{4} = 6$

So the directrix has equation $x + 6 = 0$ or $x = -6$

Therefore at X, $x = -6$

9 c Tangent at B is of the form $ty = x + 6t^2$ and passes through $X(-6, 9)$
substitute $x = -6$ and $y = 9$ into $ty = x + 6t^2$:

$$t(9) = -6 + 6t^2$$
$$0 = 6t^2 - 9t - 6$$
$$2t^2 - 3t - 2 = 0$$
$$(t-2)(2t+1) = 0$$
$$t = 2, \quad -\frac{1}{2}$$

When $t = 2$,
$x = 6(2)^2 = 24, \quad y = 12(2) = 24$
$\Rightarrow (24, 24)$

When $t = -\frac{1}{2}$,
$x = 6\left(-\frac{1}{2}\right)^2 = \frac{3}{2}, \quad y = 12\left(-\frac{1}{2}\right) = -6$
$\Rightarrow \left(\frac{3}{2}, -6\right)$

The possible coordinates of B are (24, 24) and $\left(\frac{3}{2}, -6\right)$

10 $y^2 = 16x$

$2y\frac{dy}{dx} = 16$ so $\frac{dy}{dx} = \frac{8}{y}$

At $P(4p^2, 8p)$, $\frac{dy}{dx} = \frac{8}{8p} = \frac{1}{p}$

The gradient of the normal at P is therefore $-p$

At $Q(4q^2, 8q)$, $\frac{dy}{dx} = \frac{8}{8q} = \frac{1}{q}$

The gradient of the normal at Q is therefore $-q$

The normal to P has equation

$$\frac{y - 8p}{x - 4p^2} = -p$$
$$y - 8p = -p(x - 4p^2)$$
$$y - 8p = -px + 4p^3$$
$$y = -px + 4p^3 + 8p \quad \textbf{(1)}$$

Similarly, the normal to Q has equation
$$y = -qx + 4q^3 + 8q \quad \textbf{(2)}$$

Solving the two normal equations simultaneously:

$$-px + 4p^3 + 8p = -qx + 4q^3 + 8q$$
$$x(q - p) = 4(q^3 - p^3) + 8(q - p)$$

$q \neq p$, so

$$x = \frac{4(q^3 - p^3)}{(q - p)} + 8$$
$$x = \frac{4(p^2 + pq + q^2)(q - p)}{(q - p)} + 8$$
$$x = 4(p^2 + pq + q^2) + 8$$

Substituting into **(1)**:

$$y = -4p(p^2 + pq + q^2) - 8p + 4p^3 + 8p$$
$$y = -4p^3 - 4p^2q - 4pq^2 - 8p + 4p^3 + 8p$$
$$y = -4p^2q - 4pq^2$$
$$y = -4pq(p + q)$$

Therefore P and Q meet at
$\left(8 + 4(p^2 + pq + q^2), -4pq(p + q)\right)$

11 a $xy = 64 \Rightarrow y = \frac{64}{x} = 64x^{-1}$

$\frac{dy}{dx} = -\frac{64}{x^2}$

At $P\left(8p, \frac{8}{p}\right)$:

$\frac{dy}{dx} = -\frac{64}{(8p)^2} = -\frac{64}{64p^2} = -\frac{1}{p^2}$

The equation of the tangent at P is:

$\frac{y - \frac{8}{p}}{x - 8p} = -\frac{1}{p^2}$

$p^2y - 8p = 8p - x$

$p^2y + x = 16p$ **(1)**

b Using the result from part **a**, the equation of the tangent at $Q\left(8q, \frac{8}{q}\right)$ is

$q^2y + x = 16q$ **(2)**

Solving equations **(1)** and **(2)** simultaneously to find the coordinates of R:

$16p - p^2y = 16q - q^2y$

$q^2y - p^2y = 16q - 16p$

$y(q^2 - p^2) = 16(q - p)$

$y = \frac{16(q-p)}{(q^2 - p^2)}$

$= \frac{16(q-p)}{(q+p)(q-p)}$

$= \frac{16}{(q+p)}$

Substitute into equation **(1)**:

$x = 16p - p^2y = 16p - \frac{16p^2}{(q+p)}$

Hence R has coordinates

$\left(16p - \frac{16p^2}{(q+p)}, \frac{16}{(q+p)}\right)$

So the line OR has gradient

$m_{OR} = \frac{\frac{16}{(q+p)}}{16p - \frac{16p^2}{(q+p)}}$

$= \frac{16}{16p(p+q) - 16p^2}$

$= \frac{1}{p(p+q) - p^2} = \frac{1}{pq}$

The line PQ has gradient

$m_{PQ} = \frac{\frac{8}{p} - \frac{8}{q}}{8p - 8q} = \frac{\frac{q-p}{pq}}{p-q} = -\frac{1}{pq}$

Since OR is perpendicular to PQ,

it follows that $\left(\frac{1}{pq}\right)\left(-\frac{1}{pq}\right) = -1$

So $\frac{-1}{p^2q^2} = -1$

Therefore $p^2q^2 = 1$, as required.

12 a $x = at^2$ and $y = 2at$

$\frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$

(alternatively, you can find the cartesian equation and differentiate with respect to x)

At $P(at^2, 2at)$, the equation of the tangent is:

$\frac{y - 2at}{x - at^2} = \frac{1}{t}$

$t(y - 2at) = x - at^2$

$ty - 2at^2 = x - at^2$

$ty = x + at^2$

b The tangent intersects the x-axis where $y = 0$

$\Rightarrow 0 = x + at^2$

$x = -at^2$

Therefore the coordinates of T are $T(-at^2, 0)$

12 c You have coordinates
$P(at^2, 2at)$, $S(a, 0)$ and $T(-at^2, 0)$

The gradient of PT is
$$\frac{2at-0}{at^2-(-at^2)}=\frac{2at}{2at^2}=\frac{1}{t}$$

The gradient of PS is
$$\frac{2at-0}{at^2-a}=\frac{2at}{at^2-a}=\frac{2t}{t^2-1}$$

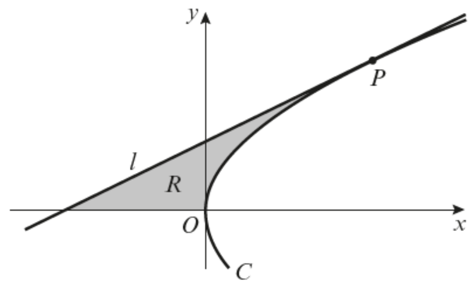
If PT is perpendicular to PS, then
$$\left(\frac{1}{t}\right)\left(\frac{2t}{t^2-1}\right)=-1$$
$$\frac{2}{t^2-1}=-1$$
$$2=1-t^2$$
$$t^2=-1$$

This is a contradiction, so PT can never be perpendicular to PS

13 a $y^2=4x$
$$2y\frac{dy}{dx}=4 \text{ so } \frac{dy}{dx}=\frac{4}{2y}=\frac{2}{y}$$
At P, $\frac{dy}{dx}=\frac{2}{2p}=\frac{1}{p}$

The equation for l is therefore
$$\frac{y-2p}{x-p^2}=\frac{1}{p}$$
$$p(y-2p)=x-p^2$$
$$py-2p^2=x-p^2$$
$$py=x+p^2$$

13 b



Suppose l intersects the x-axis at a point Q:

Substituting $y=0$ in the equation for l gives $0=x+p^2$
So $x=-p^2$ at Q
So Q has coordinates $Q(-p^2, 0)$

The shaded area is given by
$$\frac{1}{2}\times(p^2+p^2)\times 2p-\int_0^{p^2} y\,dx$$
$$=2p^3-\int_0^{p^2} 2x^{\frac{1}{2}}\,dx$$
$$=2p^3-\left[\frac{4x^{\frac{3}{2}}}{3}\right]_0^{p^2}$$
$$=2p^3-\frac{4p^3}{3}$$
$$=\frac{2p^3}{3}$$