Conic Sections 1 2E

1 a $y^2 = 4x$

$2y\dfrac{dy}{dx} = 4$ so $\dfrac{dy}{dx} = \dfrac{2}{y}$

At (16, 8), $\dfrac{dy}{dx} = \dfrac{2}{8} = \dfrac{1}{4}$

Tangent is:

$$y - 8 = \frac{1}{4}(x - 16)$$
$$4y - 32 = x - 16$$
$$0 = x - 4y - 16 + 32$$
$$x - 4y + 16 = 0$$

Therefore, the equation of the tangent is $x - 4y + 16 = 0$

b $y^2 = 8x$

$2y\dfrac{dy}{dx} = 8$ so $\dfrac{dy}{dx} = \dfrac{4}{y}$

At $\left(4, 4\sqrt{2}\right)$, $\dfrac{dy}{dx} = \dfrac{4}{4\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

Tangent is:

$$y - 4\sqrt{2} = \frac{\sqrt{2}}{2}(x - 4)$$
$$2y - 8\sqrt{2} = \sqrt{2}(x - 4)$$
$$2y - 8\sqrt{2} = \sqrt{2}x - 4\sqrt{2}$$
$$0 = \sqrt{2}x - 2y - 4\sqrt{2} + 8\sqrt{2}$$
$$\sqrt{2}x - 2y + 4\sqrt{2} = 0$$

Therefore, the equation of the tangent is $\sqrt{2}x - 2y + 4\sqrt{2} = 0$

c $xy = 25 \Rightarrow y = 25x^{-1}$

$\dfrac{dy}{dx} = -25x^{-2} = -\dfrac{25}{x^2}$

At (5, 5), $\dfrac{dy}{dx} = -\dfrac{25}{5^2} = -\dfrac{25}{25} = -1$

Tangent is:

$$y - 5 = -1(x - 5)$$
$$y - 5 = -x + 5$$
$$x + y - 5 - 5 = 0$$
$$x + y - 10 = 0$$

Equation of the tangent is $x + y - 10 = 0$

1 d $xy = 4 \Rightarrow y = 4x^{-1}$

$\dfrac{dy}{dx} = -4x^{-2} = -\dfrac{4}{x^2}$

At $x = \dfrac{1}{2}$, $\dfrac{dy}{dx} = -\dfrac{4}{\left(\frac{1}{2}\right)^2} = -\dfrac{4}{\frac{1}{4}} = -16$

When $x = \dfrac{1}{2}, y = \dfrac{4}{\left(\frac{1}{2}\right)} = 8 \Rightarrow \left(\dfrac{1}{2}, 8\right)$

Tangent at $\left(\dfrac{1}{2}, 8\right)$ is

$$y - 8 = -16\left(x - \frac{1}{2}\right)$$
$$y - 8 = -16x + 8$$
$$16x + y - 8 - 8 = 0$$
$$16x + y - 16 = 0$$

Therefore, the equation of the tangent is $16x + y - 16 = 0$

e $y^2 = 7x$

$2y\dfrac{dy}{dx} = 7$ so $\dfrac{dy}{dx} = \dfrac{7}{2y}$

At $(7, -7)$, $\dfrac{dy}{dx} = \dfrac{7}{2(-7)} = -\dfrac{1}{2}$

Tangent is:

$$y + 7 = -\frac{1}{2}(x - 7)$$
$$2y + 14 = -1(x - 7)$$
$$2y + 14 = -x + 7$$
$$x + 2y + 14 - 7 = 0$$
$$x + 2y + 7 = 0$$

Therefore, the equation of the tangent is $x + 2y + 7 = 0$

1 f $xy = 16 \Rightarrow y = 16x^{-1}$

$\frac{dy}{dx} = -16x^{-2} = -\frac{16}{x^2}$

At $x = 2\sqrt{2}$, $\frac{dy}{dx} = -\frac{16}{(2\sqrt{2})^2} = -\frac{16}{8} = -2$

When $x = 2\sqrt{2}$,

$y = \frac{16}{2\sqrt{2}} = \frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{\sqrt{2}\sqrt{2}} = 4\sqrt{2}$

Hence, tangent at $(2\sqrt{2},\ 4\sqrt{2})$ is:

$y - 4\sqrt{2} = -2(x - 2\sqrt{2})$

$y - 4\sqrt{2} = -2x + 4\sqrt{2}$

$2x + y - 4\sqrt{2} - 4\sqrt{2} = 0$

$2x + y - 8\sqrt{2} = 0$

Therefore, the equation of the tangent is $2x + y - 8\sqrt{2} = 0$

2 a Substituting $y = 10$ into $y^2 = 20x$ gives

$(10)^2 = 20x \Rightarrow x = \frac{100}{20} = 5 \Rightarrow (5, 10)$

Differentiating $y^2 = 20x$ implicitly gives

$2y\frac{dy}{dx} = 20$ so $\frac{dy}{dx} = \frac{10}{y}$

At $(5, 10)$, $\frac{dy}{dx} = \frac{10}{10} = 1$

Gradient of tangent at $(5, 10)$ is $m_T = 1$

So gradient of normal is $m_N = -1$

Normal is the line:

$y - 10 = -1(x - 5)$

$y - 10 = -x + 5$

$x + y - 10 - 5 = 0$

$x + y - 15 = 0$

Therefore, the equation of the normal is $x + y - 15 = 0$

2 b $xy = 9 \Rightarrow y = 9x^{-1}$

$\frac{dy}{dx} = -9x^{-2} = -\frac{9}{x^2}$

At $x = -\frac{3}{2}$,

$\frac{dy}{dx} = -\frac{9}{(-\frac{3}{2})^2} = -\frac{9}{(\frac{9}{4})} = -\frac{36}{9} = -4$

Gradient of tangent at $\left(-\frac{3}{2}, -6\right)$ is

$m_T = -4$

So gradient of normal is $m_N = \frac{-1}{-4} = \frac{1}{4}$

Normal is the line:

$y + 6 = \frac{1}{4}\left(x + \frac{3}{2}\right)$

$4y + 24 = x + \frac{3}{2}$

$8y + 48 = 2x + 3$

$0 = 2x - 8y + 3 - 48$

$0 = 2x - 8y - 45$

Therefore, the equation of the normal is $2x - 8y - 45 = 0$

3 a $xy = 32 \Rightarrow y = 32x^{-1}$

$\frac{dy}{dx} = -32x^{-2} = -\frac{32}{x^2}$

At $A(-2, -16)$,

$\frac{dy}{dx} = -\frac{32}{(-2)^2} = -\frac{32}{4} = -8$

Gradient of tangent at $A(-2, -16)$ is $m_T = -8$

So gradient of normal at $A(-2, -16)$ is $m_N = \frac{-1}{-8} = \frac{1}{8}$

Normal is the line:

$y + 16 = \frac{1}{8}(x + 2)$

$8y + 128 = x + 2$

$0 = x - 8y + 2 - 128$

$0 = x - 8y - 126$

The equation of the normal to H at A is $x - 8y - 126 = 0$

3 b Normal to H at A: $x-8y-126=0$ **(1)**

Hyperbola H: $xy=32$ **(2)**

Rearranging **(2)** gives $y=\frac{32}{x}$

Substituting this equation into **(1)** gives

$$x-8\left(\frac{32}{x}\right)-126=0$$
$$x-\left(\frac{256}{x}\right)-126=0$$
$$x^2-256-126x=0$$
$$x^2-126x-256=0$$
$$(x-128)(x+2)=0$$
$$x=128 \text{ or } -2$$

At A, it is already known that $x=-2$
So at B, $x=128$

Substituting $x=128$ into $y=\frac{32}{x}$ gives

$$y=\frac{32}{128}=\frac{1}{4}$$

Hence the coordinates of B are $\left(128,\frac{1}{4}\right)$

4 a The points P and Q have coordinates $P(4,12)$ and $Q(-8,-6)$

Hence gradient of PQ,

$$m_{PQ}=\frac{-6-12}{-8-4}=\frac{-18}{-12}=\frac{3}{2}$$

Hence PQ is $y-12=\frac{3}{2}(x-4)$

$$2y-24=3(x-4)$$
$$2y-24=3x-12$$
$$0=3x-2y-12+24$$
$$0=3x-2y+12$$

The line PQ has equation $3x-2y+12=0$

4 b From part **a**, the gradient of the chord PQ is $\frac{3}{2}$

The normal to H at A is parallel to the chord PQ, implies that the gradient of the normal to H at A is $\frac{3}{2}$

It follows that the gradient of the tangent to H at A is

$$m_T=\frac{-1}{m_N}=\frac{-1}{\left(\frac{3}{2}\right)}=-\frac{2}{3}$$

Now

$$H: xy=48 \Rightarrow y=48x^{-1}$$
$$\frac{dy}{dx}=-48x^{-2}=-\frac{48}{x^2}$$

At A,

$$-\frac{2}{3}=\frac{dy}{dx}$$
$$-\frac{2}{3}=-\frac{48}{x^2}$$
$$\frac{48}{x^2}=\frac{2}{3}$$
$$2x^2=144$$
$$x^2=72$$
$$x=\pm\sqrt{72}$$
$$x=\pm 6\sqrt{2}$$

When $x=6\sqrt{2}\Rightarrow$

$$y=\frac{48}{6\sqrt{2}}=\frac{8}{\sqrt{2}}=\frac{8\sqrt{2}}{\sqrt{2}\sqrt{2}}=4\sqrt{2}$$

When $x=-6\sqrt{2}\Rightarrow$

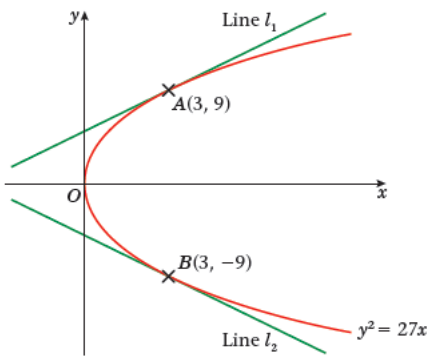
$$y=\frac{48}{-6\sqrt{2}}=\frac{-8}{\sqrt{2}}=\frac{-8\sqrt{2}}{\sqrt{2}\sqrt{2}}=-4\sqrt{2}$$

Hence the possible exact coordinates of A are $\left(6\sqrt{2},4\sqrt{2}\right)$ or $\left(-6\sqrt{2},-4\sqrt{2}\right)$

5 a Substituting $x = 3$ in the equation $y^2 = 27x$ gives $y^2 = 81$, so $y = \pm 9$

Therefore the coordinates are $A(3, 9)$ and $B(3, -9)$

b



c i $y^2 = 27x$

$$2y\frac{dy}{dx} = 27 \Rightarrow \frac{dy}{dx} = \frac{27}{2y}$$

At A, $\frac{dy}{dx} = \frac{27}{2\times 9} = \frac{27}{18} = \frac{3}{2}$

The equation of l_1 is therefore

$$\frac{y-9}{x-3} = \frac{3}{2}$$
$$2(y-9) = 3(x-3)$$
$$2y - 18 = 3x - 9$$
$$3x - 2y + 9 = 0$$

ii At B, $\frac{dy}{dx} = \frac{27}{2\times(-9)} = -\frac{27}{18} = -\frac{3}{2}$

The equation of l_2 is therefore:

$$\frac{y-(-9)}{x-3} = -\frac{3}{2}$$
$$2(y+9) = -3(x-3)$$
$$2y + 18 = -3x + 9$$
$$3x + 2y + 9 = 0$$

6 a $xy = \sqrt{3}t \times \left(\frac{\sqrt{3}}{t}\right)$

$$xy = \frac{3t}{t}$$

A Cartesian equation of H is $xy = 3$

b $xy = 3 \Rightarrow y = 3x^{-1}$

$$\frac{dy}{dx} = -3x^{-2} = -\frac{3}{x^2}$$

At $x = 2\sqrt{3}$,

$$\frac{dy}{dx} = -\frac{3}{(2\sqrt{3})^2} = -\frac{3}{12} = -\frac{1}{4}$$

Gradient of tangent at P is $m_T = -\frac{1}{4}$

So gradient of normal at P is

$$m_N = \frac{-1}{\left(-\frac{1}{4}\right)} = 4$$

At P, $x = 2\sqrt{3}$

$$\therefore 2\sqrt{3} = \sqrt{3}t$$
$$\Rightarrow t = \frac{2\sqrt{3}}{\sqrt{3}} = 2$$

When $t = 2, y = \frac{\sqrt{3}}{2} \Rightarrow P\left(2\sqrt{3}, \frac{\sqrt{3}}{2}\right)$

Normal is the line:

$$y - \frac{\sqrt{3}}{2} = 4(x - 2\sqrt{3})$$
$$2y - \sqrt{3} = 8(x - 2\sqrt{3})$$
$$2y - \sqrt{3} = 8x - 16\sqrt{3}$$
$$0 = 8x - 2y - 16\sqrt{3} + \sqrt{3}$$
$$0 = 8x - 2y - 15\sqrt{3}$$

The equation of the normal to H at P is $8x - 2y - 15\sqrt{3} = 0$

6 c Normal to H at P:

$$8x-2y-15\sqrt{3}=0 \quad \textbf{(1)}$$

Hyperbola H: $xy=3$ **(2)**

Rearranging **(2)** gives $y=\frac{3}{x}$

Substituting this equation into **(1)** gives

$$8x-2\left(\frac{3}{x}\right)-15\sqrt{3}=0$$
$$8x-\left(\frac{6}{x}\right)-15\sqrt{3}=0$$
$$8x^2-6-15\sqrt{3}x=0$$
$$8x^2-15\sqrt{3}x-6=0$$

At P, it is already known that $x=2\sqrt{3}$, so $(x-2\sqrt{3})$ is a factor of this quadratic equation.

Hence,

$$(x-2\sqrt{3})(8x+\sqrt{3})=0$$
$$\therefore x=2\sqrt{3} \quad (\text{at } P)$$
$$\text{or } x=-\frac{\sqrt{3}}{8} \quad (\text{at } Q).$$

At Q, $x=-\frac{\sqrt{3}}{8}$

$$\therefore \frac{-\sqrt{3}}{8}=\sqrt{3}t$$
$$\Rightarrow t=\frac{-\sqrt{3}}{8\sqrt{3}}=-\frac{1}{8}$$

When $t=-\frac{1}{8}$, $y=\frac{\sqrt{3}}{\left(-\frac{1}{8}\right)}=-8\sqrt{3}$

$$\Rightarrow Q\left(-\frac{1}{8}\sqrt{3},\ -8\sqrt{3}\right)$$

Hence the coordinates of Q are $\left(-\frac{1}{8}\sqrt{3},-8\sqrt{3}\right)$

7 a Substituting $x=4t^2$ and $y=8t$ into $xy=4$ gives

$$\left(4t^2\right)\left(8t\right)=4$$
$$32t^3=4$$
$$t^3=\frac{4}{32}=\frac{1}{8}$$
$$\text{So } t=\sqrt[3]{\left(\frac{1}{8}\right)}=\frac{1}{2}$$

When $t=\frac{1}{2}, x=4\left(\frac{1}{2}\right)^2=1$

When $t=\frac{1}{2}, y=8\left(\frac{1}{2}\right)=4$

Hence the value of t is $\frac{1}{2}$ and P has coordinates $(1, 4)$

b $xy=4\Rightarrow y=4x^{-1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-4x^{-2}=-\frac{4}{x^2}$$

At $P(1, 4)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{4}{(1)^2}=-\frac{4}{1}=-4$

Gradient of tangent to H at $P(1,4)$ is $m_T=-4$

So gradient of normal to H at $P(1,4)$ is $m_N=\frac{-1}{-4}=\frac{1}{4}$

Hence, normal to H at P is the line:

$$y-4=\frac{1}{4}(x-1)$$
$$4y-16=x-1$$
$$0=x-4y+15$$

Normal meets x-axis at N:

$$y=0\Rightarrow 0=x+15$$
$$\Rightarrow x=-15$$

Coordinates of N are $(-15, 0)$

7 c $y^2 = 16x$

$2y\frac{dy}{dx} = 16$ so $\frac{dy}{dx} = \frac{8}{y}$

At $P(1,4)$, $\frac{dy}{dx} = \frac{8}{4} = 2$

Tangent to C at P is:

$y - 4 = 2(x-1)$

$y - 4 = 2x - 2$

$0 = 2x - y + 2$

Tangent cuts x-axis at T:

$y = 0 \Rightarrow 0 = 2x + 2$

$\Rightarrow x = -1$

Coordinates of T are $(-1, 0)$

d **<fp1_2e _aw1>**

From the diagram,

Area $\Delta\, NPT$ = Area $(R+S)$ − Area(S)

$= \frac{1}{2}(16)(4) - \frac{1}{2}(2)(4)$

$= 32 - 4$

$= 28$

Therefore, Area $\Delta\ NPT = 28$