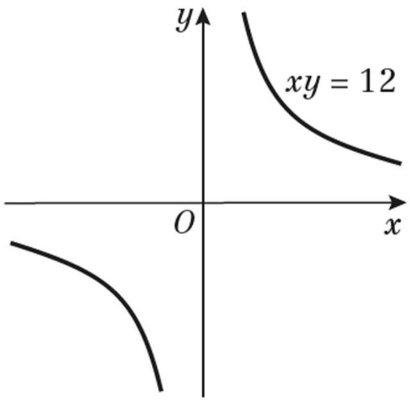
Conic Sections 1 2D

1 a



b Solving $xy = 12$ and $y = -3x + 15$ simultaneously:

$$x(-3x+15) = 12$$
$$-3x^2 + 15x = 12$$
$$x^2 - 5x + 4 = 0$$
$$(x-4)(x-1) = 0$$

So $x = 1 \Rightarrow y = \frac{12}{1} = 12$

or $x = 4 \Rightarrow y = \frac{12}{4} = 3$

So the coordinates are $P(1, 12)$ and $Q(4, 3)$

c The midpoint of PQ is given by

$$M\left(\frac{1+4}{2}, \frac{12+3}{2}\right) = M\left(\frac{5}{2}, \frac{15}{2}\right)$$

The gradient of PQ is $\frac{12-3}{1-4} = \frac{9}{-3} = -3$

The gradient of the perpendicular bisector of PQ is therefore

$$\frac{-1}{(-3)} = \frac{1}{3}$$

The equation of the bisector is therefore

$$y - \frac{15}{2} = \frac{1}{3}\left(x - \frac{5}{2}\right)$$
$$y - \frac{15}{2} = \frac{x}{3} - \frac{5}{6}$$

So $y = \frac{x}{3} + \frac{20}{3}$

1 d Solving $xy = 12$ and $y = \frac{x}{3} + \frac{20}{3}$ simultaneously:

$$x\left(\frac{x}{3} + \frac{20}{3}\right) = 12$$
$$x^2 + 20x = 36$$
$$x^2 + 20x - 36 = 0$$

Using the quadratic formula:

$$x = \frac{-20 \pm \sqrt{20^2 - 4 \times 1 \times (-36)}}{2}$$
$$x = \frac{-20 \pm \sqrt{544}}{2}$$
$$x = \frac{-20 \pm 4\sqrt{34}}{2} = -10 \pm 2\sqrt{34}$$

2 a Solving $xy = 9$ and $y = x$ simultaneously:

$$x^2 = 9$$

So $x = 3$ and $y = 3$ (point Q)
or $x = -3$ and $y = -3$ (point P)

2 b Solving $3x - y + 6 = 0$ and $xy = 9$ simultaneously:

$$3x - \frac{9}{x} + 6 = 0$$

$$3x^2 + 6x - 9 = 0$$
$$x^2 + 2x - 3 = 0$$
$$(x-1)(x+3) = 0$$

$x = -3$ at point P.

Therefore at S, $x = 1$ and $y = 9$

So S has coordinates $S(1, 9)$

Solving $x - 3y - 6 = 0$ and $xy = 9$ simultaneously:

$$x - 3\left(\frac{9}{x}\right) - 6 = 0$$
$$x^2 - 6x - 27 = 0$$
$$(x-9)(x+3) = 0$$

$x = -3$ at point P.

Therefore at T, $x = 9$ and $y = 1$

So T has coordinates $T(9, 1)$

Therefore $ST = \sqrt{(9-1)^2 + (1-9)^2}$
$= \sqrt{128} = 8\sqrt{2}$

c The midpoint of ST is
$M\left(\frac{1+9}{2}, \frac{9+1}{2}\right) = M(5, 5)$

Therefore M lies on the line $y = x$

3 $x = 6t$ and $y = \frac{6}{t}$, so $xy = 36$

Solving $xy = 36$ and $3x + 4y + 48 = 0$ simultaneously gives:

$$3x + 4\left(\frac{36}{x}\right) + 48 = 0$$
$$3x^2 + 48x + 144 = 0$$
$$x^2 + 16x + 48 = 0$$
$$(x+4)(x+12) = 0$$

When $x = -4$ then $y = \frac{36}{-4} = -9$

When $x = -12$ then $y = \frac{36}{-12} = -3$

Solving $xy = 36$ and $4x - 3y - 11 = 0$ simultaneously gives:

$$4x - 3\left(\frac{36}{x}\right) - 11 = 0$$
$$4x^2 - 11x - 108 = 0$$

Using the quadratic formula:

$$x = \frac{11 \pm \sqrt{(-11)^2 - 4 \times 4 \times (-108)}}{8}$$
$$= \frac{11 \pm 43}{8}$$

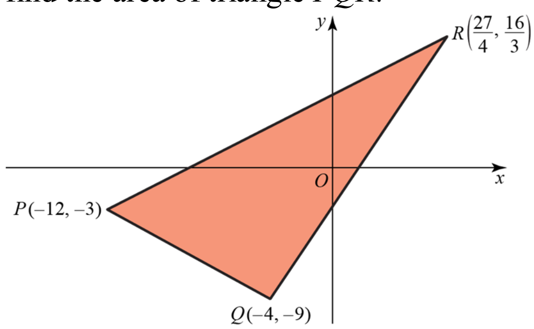
When $x = -4$ then $y = -9$ (point Q)

When $x = \frac{27}{4}$ then $y = \frac{16}{3}$ (point R)

Both the lines $3x + 4y + 48 = 0$ and $4x - 3y - 11 = 0$ intersect the parabola $xy = 36$ at the point (–4, –9), so Q is the point (–4, –9)

Hence, P is (–12, –3) and R is $\left(\frac{27}{4}, \frac{16}{3}\right)$

There are two approaches you can take to find the area of triangle PQR:



3 continued

Method 1: Using Area $= \frac{1}{2}ab\sin C$

$$|PR| = \sqrt{\left(\frac{27}{4}+12\right)^2 + \left(\frac{16}{3}+3\right)^2} = \frac{25}{12}\sqrt{97}$$

$$|QR| = \sqrt{\left(\frac{27}{4}+4\right)^2 + \left(\frac{16}{3}+9\right)^2} = \frac{215}{12}$$

$$|PQ| = \sqrt{(-12+4)^2 + (-3+9)^2} = 10$$

Cosine rule: $\cos PRQ = \dfrac{|PR|^2 + |QR|^2 - |PQ|^2}{2|PR||QR|}$

$$= \frac{\frac{60625}{144} + \frac{46225}{144} - 100}{2\left(\frac{25\sqrt{97}}{12}\right)\left(\frac{215}{12}\right)}$$

$$PRQ = 29.167^\circ$$

Area $= = \frac{1}{2}|PR||QR|\sin PRQ$

$$= \frac{1}{2}\left(\frac{25\sqrt{97}}{12}\right)\left(\frac{215}{12}\right)\sin(29.167)$$

$= 89.6$ units

Method 2: Geometrical approach

Area $= \left(18\frac{3}{4}\times 14\frac{1}{3}\right) - \frac{1}{2}\left(18\frac{3}{4}\times 8\frac{1}{3}\right)$

$- \frac{1}{2}(6\times 8) - \frac{1}{2}\left(10\frac{3}{4}\times 14\frac{1}{3}\right)$

$= \left(\frac{75}{4}\times\frac{43}{3}\right) - \frac{1}{2}\left(\frac{75}{4}\times\frac{25}{3}\right)$

$-24 - \frac{1}{2}\left(\frac{43}{4}\times\frac{43}{3}\right)$

$= \frac{3225}{12} - \frac{1}{2}\left(\frac{1875}{12}\right)$

$-24 - \frac{1}{2}\left(\frac{1849}{12}\right)$

$= \frac{3225}{12} - \frac{1875}{24} - 24 - \frac{1849}{24}$

$= \frac{6450}{24} - \frac{1875}{24} - \frac{576}{24} - \frac{1849}{24}$

$= \frac{2150}{24}$

$= \frac{1075}{12}$

4 $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$

The gradient of PQ is

$$m_{PQ} = \frac{\frac{c}{q} - \frac{c}{p}}{cq - cp}$$

$$= \frac{\frac{1}{q} - \frac{1}{p}}{q-p}$$

$$= \frac{\frac{p-q}{pq}}{q-p}$$

$$= \frac{p-q}{pq(q-p)}$$

$$= \frac{-(q-p)}{pq(q-p)}$$

$$= -\frac{1}{pq}$$

The equation of the line PQ is therefore

$$\frac{y - \frac{c}{p}}{x - cp} = -\frac{1}{pq}$$

$$pq\left(y - \frac{c}{p}\right) = -(x - cp)$$

$$pqy - cq = cp - x$$

$$x + pqy = cp + cq$$

So $x + pqy = c(p+q)$ as required.

5 Solving $y^2 = 4ax$ **(1)**
and $xy = c^2$ **(2)**
simultaneously:

From **(2)**, $y = \frac{c^2}{x} \Rightarrow y^2 = \frac{c^4}{x^2}$

Substituting into equation **(1)**:

$$\frac{c^4}{x^2} = 4ax$$
$$c^4 = 4ax^3$$
$$x^3 = \frac{c^4}{4a}$$
$$x = \frac{c^{\frac{4}{3}}}{(4a)^{\frac{1}{3}}}$$

Therefore **(2)** $\Rightarrow y = \frac{c^2}{x} = c^2 \times \frac{(4a)^{\frac{1}{3}}}{c^{\frac{4}{3}}}$

$$y = (4a)^{\frac{1}{3}} c^{\frac{2}{3}}$$

Therefore C and H intersect exactly once, and the point of intersection is $\left(\frac{c^{\frac{4}{3}}}{(4a)^{\frac{1}{3}}}, (4a)^{\frac{1}{3}} c^{\frac{2}{3}}\right)$

6 $xy = c^2$

At P, $x = \frac{c}{2}$, so $\frac{cy}{2} = c^2$

Hence $y = \frac{2c^2}{c} = 2c$

So the coordinates of P are $P\left(\frac{c}{2}, 2c\right)$

At Q, $x = -4c$, so $-4cy = c^2$

Hence $y = \frac{c^2}{-4c} = -\frac{c}{4}$

So the coordinates of Q are $Q\left(-4c, -\frac{c}{4}\right)$

Therefore length PQ

$$= \sqrt{\left(\frac{c}{2} - (-4c)\right)^2 + \left(2c - \left(-\frac{c}{4}\right)\right)^2}$$
$$= \sqrt{\left(\frac{9c}{2}\right)^2 + \left(\frac{9c}{4}\right)^2}$$
$$= \sqrt{\frac{81c^2}{4} + \frac{81c^2}{16}}$$
$$= \sqrt{\frac{405c^2}{16}} = \frac{c\sqrt{405}}{4} = \frac{9\sqrt{5}c}{4}$$

7 **a** Substitute $x = 9t$ and $y = \frac{9}{t}$ in the equation $4x - 3y + 69 = 0$:

$$4(9t) - 3\left(\frac{9}{t}\right) + 69 = 0$$
$$36t - \frac{27}{t} + 69 = 0$$
$$36t^2 + 69t - 27 = 0$$
$$12t^2 + 23t - 9 = 0$$

b $12t^2 + 23t - 9 = 0$

$(4t+9)(3t-1) = 0$

When $t = -\frac{9}{4} \Rightarrow x = -\frac{81}{4}$ and $y = -4$

When $t = \frac{1}{3} \Rightarrow x = 3$ and $y = 27$

Therefore the required points are $\left(-\frac{81}{4}, -4\right)$ and $(3, 27)$

8 **a** $x = 12t$ and $y = \frac{12}{t}$, so $xy = 144$

b When $t = \frac{1}{2}$, then $x = 6$ and $y = 24$, so the coordinate of P is $P(6, 24)$

When $t = 6$, then $x = 72$ and $y = 2$, so the coordinate of Q is $Q(72, 2)$

$$|PQ| = \sqrt{(72-6)^2 + (2-24)^2}$$
$$= \sqrt{66^2 + 22^2}$$
$$= \sqrt{4840}$$
$$= 22\sqrt{10}$$

8 c PQ has midpoint

$$M\left(\frac{6+72}{2}, \frac{24+2}{2}\right) = M(39, 13)$$

The gradient of PQ is

$$\frac{2-24}{72-6} = -\frac{22}{66} = -\frac{1}{3}$$

The gradient of the perpendicular bisector of PQ is therefore 3

The equation of the perpendicular bisector of PQ is therefore:

$$\frac{y-13}{x-39} = 3$$

$$y-13 = 3(x-39)$$

$$y-13 = 3x-117$$

$$y = 3x-104$$

9 a Solving $xy = 8$ and $x+2y-10=0$ simultaneously:

$$x+2\left(\frac{8}{x}\right)-10=0$$

$$x^2-10x+16=0$$

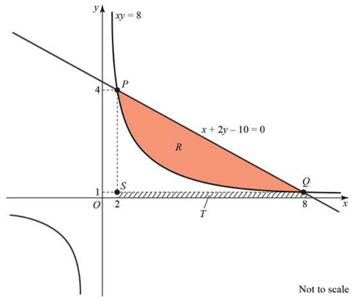
$$(x-2)(x-8)=0$$

$$x=2 \Rightarrow y=4$$

$$x=8 \Rightarrow y=1$$

Therefore the required coordinates are $P(2, 4)$ and $Q(8, 1)$

9 b



As shown in the diagram,

Area R = (Area of triangle *PQS*)
+ (Area of rectangle *T*)
– (Area under $xy = 8$ between $x = 8$ and $x = 2$)

$$R = \left(\frac{1}{2}\times 3\times 6\right) + (6\times 1) - \int_2^8 \frac{8}{x}\,dx$$

$$= 15 - 8[\ln x]_2^8$$

$$= 15 - (8\ln 8 - 8\ln 2)$$

$$= 15 - 8\ln 4$$

Challenge

C is a hyperbola of the form $xy = c^2$ rotated through an angle 45° anticlockwise about the origin.

The transformation matrix required is:

$$\begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Applying this to a general point on the hyperbola $\left(cp, \frac{c}{p}\right)$:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} cp \\ \frac{c}{p} \end{pmatrix} = \begin{pmatrix} \frac{cp}{\sqrt{2}} - \frac{c}{p\sqrt{2}} \\ \frac{cp}{\sqrt{2}} + \frac{c}{p\sqrt{2}} \end{pmatrix}$$

$$x^2 = \left(\frac{cp}{\sqrt{2}} - \frac{c}{p\sqrt{2}}\right)^2 = \frac{c^2p^2}{2} - c^2 + \frac{c^2}{2p^2}$$

$$y^2 = \left(\frac{cp}{\sqrt{2}} + \frac{c}{p\sqrt{2}}\right)^2 = \frac{c^2p^2}{2} + c^2 + \frac{c^2}{2p^2}$$

So $y^2 - x^2 = 2c^2$

So $k^2 = 2c^2$ and $k = c\sqrt{2}$