Conic Sections 1 2C

1 Line: $y = 2x - 3$ **(1)**

Parabola: $y^2 = 3x$ **(2)**

Substituting **(1)** into **(2)** gives

$$(2x-3)^2 = 3x$$
$$(2x-3)(2x-3) = 3x$$
$$4x^2 - 12x + 9 = 3x$$
$$4x^2 - 15x + 9 = 0$$
$$(x-3)(4x-3) = 0$$
$$x = 3 \text{ or } \frac{3}{4}$$

When $x = 3, y = 2(3) - 3 = 3$

When $x = \frac{3}{4}, y = 2\left(\frac{3}{4}\right) - 3 = -\frac{3}{2}$

Hence the coordinates of P and Q are $(3,\ 3)$ and $\left(\frac{3}{4}, -\frac{3}{2}\right)$ respectively.

2 Line: $y = x + 6$ **(1)**

Parabola: $y^2 = 32x$ **(2)**

Substituting **(1)** into **(2)** gives

$$(x+6)^2 = 32x$$
$$(x+6)(x+6) = 32x$$
$$x^2 + 12x + 36 = 32x$$
$$x^2 - 20x + 36 = 0$$
$$(x-2)(x-18) = 0$$
$$x = 2 \text{ or } 18$$

When $x = 2,\ y = 2 + 6 = 8$

When $x = 18, y = 18 + 6 = 24$

Hence the coordinates of A and B are $(2, 8)$ and $(18, 24)$ respectively.

$$|AB| = \sqrt{(18-2)^2 + (24-8)^2}$$
$$= \sqrt{16^2 + 16^2}$$
$$= 16\sqrt{2}$$

Hence the exact length AB is $16\sqrt{2}$

3 Line: $y = x - 20$ **(1)**

Parabola: $y^2 = 10x$ **(2)**

Substituting **(1)** into **(2)** gives

$$(x-20)^2 = 10x$$
$$(x-20)(x-20) = 10x$$
$$x^2 - 40x + 400 = 10x$$
$$x^2 - 50x + 400 = 0$$
$$(x-10)(x-40) = 0$$
$$x = 10 \text{ or } 40$$

When $x = 10, y = 10 - 20 = -10$

When $x = 40, y = 40 - 20 = 20$

Hence the coordinates of A and B are $(10,\ -10)$ and $(40, 20)$ respectively.

The midpoint of A and B is

$$\left(\frac{10+40}{2}, \frac{-10+20}{2}\right) = (25, 5)$$

Hence the coordinates of M are $(25, 5)$

4 a $y = 12t \Rightarrow t = \frac{y}{12}$

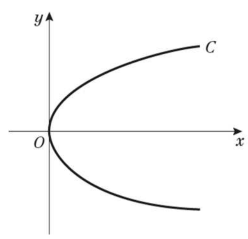
Substituting $t = \frac{y}{12}$ into $x = 6t^2$ gives

$$x = 6\left(\frac{y}{12}\right)^2$$
$$y^2 = 24x$$

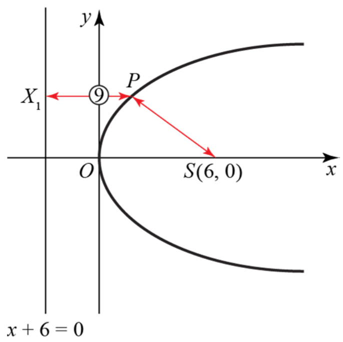
Since a general parabola has equation $y^2 = 4ax$, here $4a = 24$ so $a = \frac{24}{4} = 6$

So the focus S, has coordinates (6, 0) and the directrix has equation $x + 6 = 0$

b



4 c The distance PS is the same as the distance from P to the directrix, by the focus-directrix property.



Hence the distance $PS = 9$

d Using diagram in **c**, the x-coordinate of P and Q is $x = 9 - 6 = 3$

When $x = 3, y^2 = 24(3) = 72$

Hence $y = \pm\sqrt{72}$
$= \pm\sqrt{36}\sqrt{2}$
$= \pm 6\sqrt{2}$

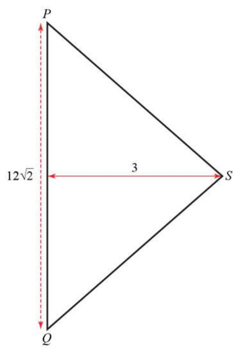
So the coordinates are of P and Q are $(3, 6\sqrt{2})$ and $(3, -6\sqrt{2})$

As P and Q are vertically above each other then

$$PQ = 6\sqrt{2} - -6\sqrt{2}$$
$$= 12\sqrt{2}$$

Hence, the distance PQ is $12\sqrt{2}$

4 e Drawing a diagram of the triangle PQS gives:



The x-coordinate of P and Q is 3 and the x-coordinate of S is 6

Hence the height of the triangle is height $= 6 - 3 = 3$

The length of the base is $12\sqrt{2}$

$$\text{Area} = \frac{1}{2}(12\sqrt{2})(3)$$
$$= \frac{1}{2}(36\sqrt{2})$$
$$= 18\sqrt{2}$$

Therefore the area of the triangle is $18\sqrt{2}$, where $k = 18$

5 a $P\left(\frac{5}{4}t^2, \frac{5}{2}t\right)$

Substituting $x = \frac{5}{4}t^2$ and $y = \frac{5}{2}t$ into $y^2 = 4ax$ gives:

$$\left(\frac{5}{2}t\right)^2 = 4a\left(\frac{5}{4}t^2\right)$$
$$\frac{25t^2}{4} = 5at^2$$
$$\frac{25}{4} = 5a \Rightarrow \frac{5}{4} = a$$

When $a = \frac{5}{4}, y^2 = 4\left(\frac{5}{4}\right)x \Rightarrow y^2 = 5x$

The Cartesian equation of C is $y^2 = 5x$

5 b When $y = 5$,

$(5)^2 = 5x$

$\frac{25}{5} = x \Rightarrow x = 5$

The x-coordinate of P is 5

c As $a = \frac{5}{4}$, the equation of the directrix of C is $x + \frac{5}{4} = 0$ or $x = -\frac{5}{4}$

Therefore the coordinates of Q are $\left(-\frac{5}{4}, 3\right)$

d The coordinates of P and Q are $(5, 5)$ and $\left(-\frac{5}{4}, 3\right)$ respectively.

$m_{PQ} = \frac{3-5}{-\frac{5}{4}-5} = \frac{-2}{-\frac{25}{4}} = \frac{8}{25}$

$l: y - 5 = \frac{8}{25}(x-5)$

$l: 25y - 125 = 8(x-5)$

$l: 25y - 125 = 8x - 40$

$l: 0 = 8x - 25y - 40 + 125$

$l: 0 = 8x - 25y + 85$

An equation for l is $8x - 25y + 85 = 0$

6 a A general parabola with equation $y^2 = 4ax$ has focus at $(a, 0)$

Here $y^2 = 4x \Rightarrow 4a = 4$, giving $a = \frac{4}{4} = 1$

So the focus S, has coordinates $(1, 0)$

b Substituting $y = 4$ into $y^2 = 4x$ gives:

$16 = 4x \Rightarrow x = \frac{16}{4} = 4$

The x-coordinate of P is 4

6 c The line l goes through S $(1, 0)$ and P $(4, 4)$

Hence gradient of l, $m_{SP} = \frac{4-0}{4-1} = \frac{4}{3}$

So l has equation:

$y - 0 = \frac{4}{3}(x-1)$

$3y = 4(x-1)$

$3y = 4x - 4$

$0 = 4x - 3y - 4$

The line l has equation $4x - 3y - 4 = 0$

d Line l: $4x - 3y - 4 = 0$ **(1)**

Parabola C: $y^2 = 4x$ **(2)**

Substituting **(2)** into **(1)** gives

$y^2 - 3y - 4 = 0$

$(y-4)(y+1) = 0$

$y = 4$ or -1

You already know that $y = 4$ at P.

So at Q, $y = -1$

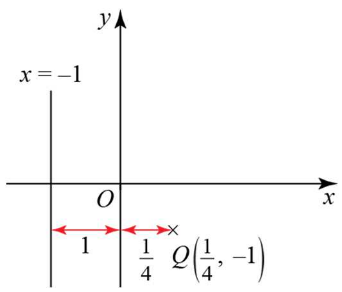
Substituting $y = -1$ into $y^2 = 4x$ gives

$(-1)^2 = 4x \Rightarrow x = \frac{1}{4}$

Hence the coordinates of Q are $\left(\frac{1}{4}, -1\right)$

e The directrix of C has equation $x + 1 = 0$ or $x = -1$

Q has coordinates $\left(\frac{1}{4}, -1\right)$



From the diagram, distance $= 1 + \frac{1}{4} = \frac{5}{4}$

Therefore the distance of the directrix of C to the point Q is $\frac{5}{4}$

7 a A general parabola with equation $y^2 = 4ax$ has focus at $(a, 0)$

Here $y^2 = 12x$ so $4a = 12 \Rightarrow a = \frac{12}{4} = 3$

Hence the focus S has coordinates (3, 0) and an directrix of C has equation: $x + 3 = 0$ or $x = -3$

The coordinates of R are $(-3, 0)$ as R lies on both the directrix and the x-axis.

b The directrix has equation $x = -3$

The (shortest) distance from P to the directrix is the distance PQ, since PQ is horizontal and hence meets the directrix at a right angle.

The distance $SP = 12$

The focus-directrix property implies that $SP = PQ = 12$

Therefore the x-coordinate of P is $x_P = PQ - 3 = 12 - 3 = 9$

As P lies on C, when $x = 9$

$y^2 = 12(9) \Rightarrow y^2 = 108$

As $y > 0$ at P, so $y_P = \sqrt{108}$

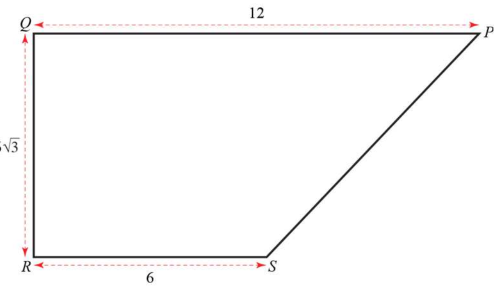
$$= \sqrt{36}\sqrt{3}$$
$$= 6\sqrt{3} \Rightarrow P(9, 6\sqrt{3})$$

Q lies on the directrix so its x-coordinate is $x_Q = -3$

Also, QP is horizontal so the y-coordinate of Q is $y_Q = y_P = 6\sqrt{3}$

Hence the exact coordinates of P are $(9, 6\sqrt{3})$ and the coordinates of Q are $(-3, 6\sqrt{3})$

7 c



$$\text{Area}\,(PQRS) = \frac{1}{2}(6+12)6\sqrt{3}$$
$$= \frac{1}{2}(18)(6\sqrt{3})$$
$$= (9)(6\sqrt{3})$$
$$= 54\sqrt{3}$$

The area of the quadrilateral $PQRS$ is $54\sqrt{3}$ and $k = 54$

8 a $P(16, 8)$

Substituting $x = 16$ and $y = 8$ into

$y^2 = 4ax$ gives $(8)^2 = 4a(16)$

$$64 = 64a$$
$$a = \frac{64}{64} = 1$$

$Q(4, b)$

Substituting $x = 4, y = b$ and $a = 1$ into

$y^2 = 4ax$ gives $b^2 = 4(1)(4)$

$$b^2 = 16$$
$$b = \pm\sqrt{16}$$
$$b = \pm 4$$

As $b < 0, b = -4$

Hence, $a = 1, b = -4$

8 b The coordinates of P and Q are $(16, 8)$ and $(4, -4)$ respectively.

$$\therefore m_{PQ} = \frac{-4-8}{4-16} = \frac{-12}{-12} = 1$$

So l has equation: $y - 8 = 1(x - 16)$

$$y = x - 8$$

c R has coordinates

$$\left(\frac{16+4}{2}, \frac{8+(-4)}{2}\right) = (10, 2)$$

d As l_2 is perpendicular to l, $m_{l_2} = -1$

Since l_2 passes through R, it has equation

$$y - 2 = -1(x - 10)$$
$$y - 2 = -x + 10$$
$$y = -x + 12$$

$\therefore l_2$ has equation $y = -x + 12$

e Line l_2: $\quad y = -x + 12 \quad$ **(1)**

Parabola C: $\quad y^2 = 4x \quad$ **(2)**

Substituting **(1)** into **(2)** gives

$$(-x+12)^2 = 4x$$
$$x^2 - 12x - 12x + 144 = 4x$$
$$x^2 - 28x + 144 = 0$$
$$(x-14)^2 - 196 + 144 = 0$$
$$(x-14)^2 - 52 = 0$$
$$(x-14)^2 = 52$$
$$x - 14 = \pm\sqrt{52}$$
$$x - 14 = \pm\sqrt{4}\sqrt{13}$$
$$x - 14 = \pm 2\sqrt{13}$$
$$x = 14 \pm 2\sqrt{13}$$

l_2 meets C at the points with x coordinates $x = 14 \pm 2\sqrt{13}$

9 a The point P is $P(at^2, 2at)$ and the focus S is $S(a, 0)$

The gradient of l is therefore

$$m_{PS} = \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2 - 1}$$

b Suppose point Q has the coordinate $Q(at_1^2, 2at_1)$

Since l passes through P, Q and S, the gradient of PS = the gradient of SQ.

Therefore

$$\frac{2t}{t^2-1} = \frac{2t_1}{t_1^2-1}$$
$$2t(t_1^2 - 1) = 2t_1(t^2 - 1)$$
$$tt_1^2 - t = t_1t^2 - t_1$$
$$tt_1^2 - t - t_1t^2 + t_1 = 0$$
$$t_1^2 + \frac{t_1}{t} - t_1t - 1 = 0$$
$$(t_1 - t)\left(t_1 + \frac{1}{t}\right) = 0$$

So $t_1 = t$ (at P)

or $t_1 = -\frac{1}{t}$ (at Q)

The coordinates of Q are therefore

$$Q(at_1^2, 2at_1) = Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$$

10 The shaded area is given by

$$\int_0^{10} y \, dx = \int_0^{10} 6\sqrt{x} \, dx$$
$$= \int_0^{10} 6x^{\frac{1}{2}} \, dx$$
$$= \left[4x^{\frac{3}{2}}\right]_0^{10}$$
$$= 4 \times 10^{\frac{3}{2}}$$
$$= 4\sqrt{1000}$$
$$= 40\sqrt{10}$$

11 At point P,

$$\frac{1}{2}x = \left(\frac{1}{8}x\right)^2$$
$$\frac{1}{2}x = \frac{x^2}{64}$$
$$32x = x^2$$
$$x^2 - 32x = 0$$
$$x(x-32) = 0$$

$x \neq 0$ since $x = 0$ at O, so $x = 32$

and $y^2 = \frac{1}{2}\times 32 = 16$, so $y = 4$

Point P is therefore $P(32,\ 4)$

Now find the equation of C above the x-axis between $x = 0$ and $x = 32$:

$$y^2 = \frac{1}{2}x, \text{ so } y = \frac{\sqrt{x}}{\sqrt{2}}$$

The shaded area is found by subtracting the area of the triangle [with vertices at O, P and (32, 0)] from the area under the curve (above the x-axis) between O and P.

$$\text{Area } R = \frac{1}{\sqrt{2}}\int_0^{32} x^{\frac{1}{2}}\,\mathrm{d}x - \frac{1}{2}(32)(4)$$
$$= \frac{1}{\sqrt{2}}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^{32} - 64$$
$$= \frac{\sqrt{2}}{3}\left(32^{\frac{3}{2}}\right) - 64$$
$$= \frac{\sqrt{2}}{3}\left(2^{\frac{3}{2}}\right)\left(16^{\frac{3}{2}}\right) - 64$$
$$= \frac{4}{3}(64) - 64$$
$$= \frac{64}{3}$$

12 a At P and Q, $y^2 = 8\times 2 = 16 \Rightarrow y = \pm 4$
Now by the diagram, $a > 0$ and $b < 0$
Therefore $a = 4$ and $b = -4$

12 b A general parabola with equation $y^2 = 4\alpha x$ has focus at $(\alpha, 0)$
Here, $y^2 = 8x \Rightarrow \alpha = 2$

The parabola has directrix $x = -\alpha \Rightarrow x = -2$
So the coordinates of T and P are $T(-2,\ 0)$ and $P(2,\ 4)$

The gradient of l is $m_l = \frac{4-0}{2-(-2)} = 1$

$\therefore$ The equation of l is $y - 4 = x - 2$
$\Rightarrow\ y = x + 2$

c The area of triangle PQT is $\frac{1}{2}\times 8\times 4 = 16$

The area bounded by the parabola, the line $x = 0$ and the line $x = 2$ is

$$2\int_2^2 \sqrt{8}\sqrt{x}\,\mathrm{d}x = 4\sqrt{2}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^2$$
$$= 4\sqrt{2}\left(\frac{4\sqrt{2}}{3}\right)$$
$$= \frac{32}{3}$$

The required area R is therefore

$$16 - \frac{32}{3} = \frac{16}{3}$$

13 a A general parabola with equation $y^2 = 4ax$ has focus at $(a, 0)$
Here, $y^2 = 16x \Rightarrow a = 4$
The focus is at the point $S(4, 0)$

b At P, $y = 4$ so $4^2 = 16x$ and $x = 1$
Therefore x-coordinate of P is $x = 1$

c l passes through $S(4,\ 0)$ and $P(1,\ 4)$
The gradient of l is $m_l = \frac{4-0}{1-4} = -\frac{4}{3}$
$\therefore$ The equation of l is

$$y = -\frac{4}{3}(x-4) \text{ or } y = -\frac{4x}{3} + \frac{16}{3}$$

13 d Q lies on both l and C, so satisfies both

$y^2 = 16x$ **(1)**

$y = -\frac{4x}{3} + \frac{16}{3}$ **(2)**

Hence, at Q

$$\left(-\frac{4x}{3} + \frac{16}{3}\right)^2 = 16x$$
$$\frac{16x^2}{9} + \frac{256}{9} - \frac{128x}{9} = 16x$$
$$16x^2 + 256 - 128x = 144x$$
$$16x^2 - 272x + 256 = 0$$
$$x^2 - 17x + 16 = 0$$
$$(x-1)(x-16) = 0$$

$x \neq 1$ (since $x = 1$ at P), so $x = 16$
So Q has x-coordinate $x = 16$

The branch of C which lies under the x-axis has equation $y = -4\sqrt{x}$

Hence, at Q, $y = -4\sqrt{16} = -16$

Defining T to be the point (16, 0), the shaded area R is therefore given by

$$\text{Area } R = \left|\int_0^{16} y \, dx\right| - [\text{area triangle } STQ]$$
$$= \left|\int_0^{16} -4\sqrt{x} \, dx\right| - \frac{1}{2} \times 12 \times 16$$
$$= \int_0^{16} 4x^{\frac{1}{2}} \, dx - 96$$
$$= 4\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^{16} - 96$$
$$= 4 \times \left(\frac{128}{3}\right) - 96$$
$$= \frac{224}{3}$$