Conic Sections 1 2B

1 The focus and directrix of a parabola with equation $y^2 = 4ax$, are $(a, 0)$ and $x + a = 0$ respectively.

a Focus (5, 0) and equation $x + 5 = 0$

So $a = 5$ and $y^2 = 4(5)x$

Hence parabola has directrix $y^2 = 20x$

b Focus (8, 0) and directrix $x + 8 = 0$

So $a = 8$ and $y^2 = 4(8)x$

Hence parabola has equation $y^2 = 32x$

c Focus (1, 0) and directrix $x = -1$ giving $x + 1 = 0$

So $a = 1$ and $y^2 = 4(1)x$

Hence parabola has equation $y^2 = 4x$

d Focus $\left(\frac{3}{2}, 0\right)$ and directrix $x = -\frac{3}{2}$ giving $x + \frac{3}{2} = 0$

So $a = \frac{3}{2}$ and $y^2 = 4\left(\frac{3}{2}\right)x$

Hence parabola has equation $y^2 = 6x$

e Focus $\left(\frac{\sqrt{3}}{2}, 0\right)$ and directrix $x + \frac{\sqrt{3}}{2} = 0$

So $a = \frac{\sqrt{3}}{2}$ and $y^2 = 4\left(\frac{\sqrt{3}}{2}\right)x$

Hence parabola has equation $y^2 = 2\sqrt{3}x$

2 The focus and directrix of a parabola with equation $y^2 = 4ax$, and $(a, 0)$ and $x + a = 0$ respectively.

a $y^2 = 12x$. So $4a = 12$, gives $a = \frac{12}{4} = 3$

So the focus has coordinates (3, 0) and the directrix has equation $x + 3 = 0$

2 b $y^2 = 20x$

So $4a = 20$, gives $a = \frac{20}{4} = 5$

So the focus has coordinates (5, 0) and the directrix has equation $x + 5 = 0$

c $y^2 = 10x$

So $4a = 10$, gives $a = \frac{10}{4} = \frac{5}{2}$

So the focus has coordinates $\left(\frac{5}{2}, 0\right)$ and the directrix has equation $x + \frac{5}{2} = 0$

d $y^2 = 4\sqrt{3}x$

So $4a = 4\sqrt{3}$, gives

$a = \frac{4\sqrt{3}}{4} = \sqrt{3}$

So the focus has coordinates $(\sqrt{3}, 0)$ and the directrix has equation $x + \sqrt{3} = 0$

e $y^2 = \sqrt{2}x$

So $4a = \sqrt{2}$, gives $a = \frac{\sqrt{2}}{4}$

So the focus has coordinates $\left(\frac{\sqrt{2}}{4}, 0\right)$ and the directrix has equation $x + \frac{\sqrt{2}}{4} = 0$

f $y^2 = 5\sqrt{2}x$

So $4a = 5\sqrt{2}$, gives $a = \frac{5\sqrt{2}}{4}$

So the focus has coordinates $\left(\frac{5\sqrt{2}}{4}, 0\right)$ and the directrix has equation

$x + \frac{5\sqrt{2}}{4} = 0$

3 a The curve with general point $(6t^2, 12t)$ has parametric equations $x = 6t^2, y = 12t$

The general parabola has parametric equations $x = at^2, y = 2at$

Comparing the two sets of equations, you see that $a = 6$

Therefore the focus is at point $S(6,0)$ and the equation of the parabola is $y^2 = 4ax$ or $y^2 = 24x$

b The curve with general point $(3\sqrt{2}t^2, 6\sqrt{2}t)$ has parametric equations $x = 3\sqrt{2}t^2, y = 6\sqrt{2}t$

The general parabola has parametric equations $x = at^2, y = 2at$

Comparing the two sets of equations, you see that $a = 3\sqrt{2}$

Therefore the focus is at point $S(3\sqrt{2}, 0)$ and the equation of the parabola is $y^2 = 4ax$, or $y^2 = 12\sqrt{2}x$

Challenge

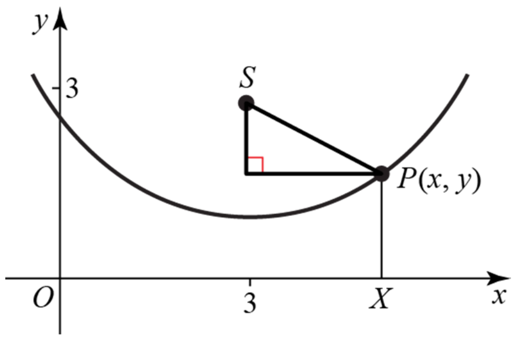
1 a A parabola $y^2 = 4ax$ has focus $S(a,0)$ and directrix $x = -a$

Therefore, a parabola $x^2 = 4ay$ has focus $S(0,a)$ and directrix $y = -a$

Therefore $a = 4$ and the parabola has equation $x^2 = 16y$

Challenge

1 b Consider a general point $P(x, y)$ on the parabola.



By the focus-directrix property, $PS = PX$.

Therefore

$$(PS)^2 = (PX)^2$$

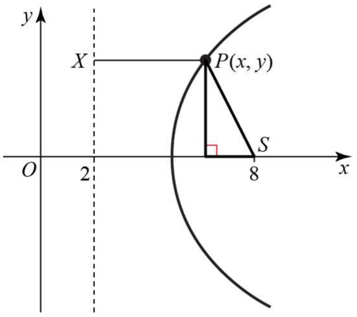
Pythagoras' theorem gives:

$$(x-3)^2 + (3-y)^2 = y^2$$
$$(x-3)^2 = y^2 - (3-y)^2$$
$$(x-3)^2 = y^2 - (9 + y^2 - 6y)$$
$$(x-3)^2 = 6y - 9$$

The equation of the parabola is therefore $(x-3)^2 = 6y - 9$

Challenge

1 c Consider a general point $P(x, y)$ on the parabola.



By the focus-directrix property, $PS = PX$.

Therefore

$$(PS)^2 = (PX)^2$$
$$(8-x)^2 + y^2 = (x-2)^2$$
$$x^2 - 16x + 64 + y^2 = x^2 - 4x + 4$$
$$y^2 = 12x - 60$$

The equation of the parabola is therefore $y^2 = 12x - 60$

Challenge

2 C is a parabola of the form $y^2 = 4ax$, rotated through an angle $\frac{\pi}{4}$ anticlockwise about the origin.

The distance between the origin and $(2, 2)$ is $2\sqrt{2}$

The transformation matrix required is

$$\begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Applying this to a general point on the parabola $(2\sqrt{2}t^2, 4\sqrt{2}t)$:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2\sqrt{2}t^2 \\ 4\sqrt{2}t \end{pmatrix} = \begin{pmatrix} 2t^2 - 4t \\ 2t^2 + 4t \end{pmatrix}$$

Now:

$$x + y = (2t^2 - 4t) + (2t^2 + 4t) = 4t^2$$

Also:

$$\frac{1}{16}(x-y)^2 = \frac{1}{16}\left(2t^2 - 4t - (2t^2 + 4t)\right)^2$$
$$= \frac{1}{16}(-8t)^2$$
$$= \frac{64t^2}{16}$$
$$= 4t^2$$

Therefore the Cartesian equation for C is given by $x + y = \frac{1}{16}(x-y)^2$