Vectors Mixed Exercise 1

1 a $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 2\\ 3 & 2 & -4\end{vmatrix}=4\mathbf{i}+10\mathbf{j}+8\mathbf{k}$

b Using the result for $\mathbf{b}\times\mathbf{c}$ from part **a** gives

$\mathbf{a}.(\mathbf{b}\times\mathbf{c})=(2\mathbf{i}+3\mathbf{j}).(4\mathbf{i}+10\mathbf{j}+8\mathbf{k})=8+30=38$

c Area of triangle OBC $=\frac{1}{2}|\mathbf{b}||\mathbf{c}|\sin\theta=\frac{1}{2}|\mathbf{b}\times\mathbf{c}|$

$$=\frac{1}{2}\left|4\mathbf{i}+10\mathbf{j}+8\mathbf{k}\right|==\frac{1}{2}\sqrt{4^2+10^2+8^2}$$

$$=\frac{1}{2}\sqrt{16+100+64}=\frac{\sqrt{180}}{2}$$

$$=\frac{3\sqrt{20}}{2}=\frac{3\sqrt{4}\sqrt{5}}{2}=3\sqrt{5}$$

d Using the result for $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$ from part **b** gives

Volume of tetrahedron $OABC$ $=\frac{1}{6}\mathbf{a}.(\mathbf{b}\times\mathbf{c})=\frac{38}{6}=\frac{19}{3}$

2 a $\overrightarrow{OB}\times\overrightarrow{OC}=(\mathbf{i}-4\mathbf{j}-3\mathbf{k})\times(-\mathbf{i}+3\mathbf{j}-\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -4 & -3\\ -1 & 3 & -1\end{vmatrix}=13\mathbf{i}+4\mathbf{j}-\mathbf{k}$

b Volume in design units

$$\frac{1}{6}\left|\overrightarrow{OA}.\left(\overrightarrow{OB}\times\overrightarrow{OC}\right)\right|=\frac{1}{6}\left|(2\mathbf{i}+\mathbf{j}+3\mathbf{k}).(13\mathbf{i}+4\mathbf{j}-\mathbf{k})\right|=\frac{1}{6}\left|26+4-3\right|=\frac{27}{6}=\frac{9}{2}$$

One cubic design unit $=4\times4\times4=64\text{ cm}^3$

So volume of prototype package $=\frac{9}{2}\times64=288\text{ cm}^3$

3 Volume of the parallelepiped is $\left|\overrightarrow{EA}.\left(\overrightarrow{EC}\times\overrightarrow{EF}\right)\right|$

Volume of the tetrahedron is $\frac{1}{6}\left|\overrightarrow{EA}.\left(\overrightarrow{EC}\times\overrightarrow{EM}\right)\right|$

$\overrightarrow{EA}=-3\mathbf{i}+\mathbf{j}-2\mathbf{k}$

$\overrightarrow{EC}=\overrightarrow{EA}+\overrightarrow{AC}=-3\mathbf{i}+\mathbf{j}-2\mathbf{k}+4\mathbf{i}+\mathbf{j}-2\mathbf{k}=\mathbf{i}+2\mathbf{j}-4\mathbf{k}$

$\overrightarrow{EF}=\overrightarrow{EA}+\overrightarrow{AF}=-3\mathbf{i}+\mathbf{j}-2\mathbf{k}+2\mathbf{i}-5\mathbf{j}+\mathbf{k}=-\mathbf{i}-4\mathbf{j}-\mathbf{k}$

$\overrightarrow{EM}=\frac{2}{3}\left(\overrightarrow{EF}\right)=-\frac{2}{3}\mathbf{i}-\frac{8}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$

So volume of parallelepiped $=\begin{vmatrix}-3 & -1 & -2\\ 1 & 2 & -4\\ -1 & -4 & -1\end{vmatrix}=-3(-18)-(-1)(-5)+(-2)(-2)=53$

And volume of tetrahedron $=\frac{1}{6}\begin{vmatrix}-3 & -1 & -2\\ 1 & 2 & -4\\ -\frac{2}{3} & -\frac{8}{3} & -\frac{2}{3}\end{vmatrix}=\frac{1}{6}\left(-3\left(-\frac{36}{3}\right)-(-1)\left(-\frac{10}{3}\right)+(-2)\left(-\frac{4}{3}\right)\right)$

$=\frac{108-10+8}{18}=\frac{106}{18}=\frac{53}{9}$

Therefore volume of the tetrahedron is $\frac{1}{9}$ of volume of the parallelepiped.

Alternatively, prove the general result as follows:

Volume of tetrahedron $=\frac{1}{6}\left|\overrightarrow{EA}.\left(\overrightarrow{EC}\times\overrightarrow{EM}\right)\right|=\frac{1}{6}\left|\overrightarrow{EA}.\left(\overrightarrow{EC}\times\frac{2}{3}\overrightarrow{EF}\right)\right|$

$=\frac{1}{6}\times\frac{2}{3}\left|\overrightarrow{EA}.\left(\overrightarrow{EC}\times\overrightarrow{EF}\right)\right|=\frac{1}{9}\left|\overrightarrow{EA}.\left(\overrightarrow{EC}\times\overrightarrow{EF}\right)\right|$

$=\frac{1}{9}$ volume of parallelepiped.

4 Let the position vector of point C relative to the origin be $\mathbf{c} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Then the volume of the tetrahedron is given by $\frac{1}{6}|\mathbf{c}.(\mathbf{a}\times\mathbf{b})|$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & 0 \\ 2 & -1 & -3 \end{vmatrix} = -6\mathbf{i} + 15\mathbf{j} - 9\mathbf{k}$$

This gives

$$\frac{1}{6}|\mathbf{c}.(\mathbf{a}\times\mathbf{b})| = \frac{1}{6}|(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).(-6\mathbf{i}+15\mathbf{j}-9\mathbf{k})| = \frac{1}{6}|-6x+15y-9z| = \frac{1}{2}|-2x+5y-3z|$$

So if the volume is 5 m^3, then the locus of admissible points is

$$\frac{1}{2}|-2x+5y-3z| = 5 \Rightarrow |-2x+5y-3z| = 10$$

So Cartesian equations satisfying this equation are

$-2x+5y-3z = 10 \Rightarrow 2x-5y+3x+10 = 0$

and $2x-5y+3z = 10 \Rightarrow 2x-5y+3x-10 = 0$

5 a Equation of L_1 is $\mathbf{r} = 3\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + s(\mathbf{j} + 2\mathbf{k})$

When $s = 2$, $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, so P lies on L_1

Equation of L_2 is $\mathbf{r} = 8\mathbf{i} + 3\mathbf{j} + t(5\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$

When $t = -1$, $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, so P lies on L_2

b $$\mathbf{b}_1\times\mathbf{b}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 2 \\ 5 & 4 & -2 \end{vmatrix} = -10\mathbf{i} + 10\mathbf{j} - 5\mathbf{k}$$

c The normal to the plane is in direction of $\mathbf{b}_1\times\mathbf{b}_2$. So $-2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ is a normal to the plane.

Using $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$, with $\mathbf{n} = 3\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{a} = 3\mathbf{i}-3\mathbf{j}-2\mathbf{k}$ (note $\mathbf{a}$ can be the position vector of any point on the plane), this gives a vector equation of the plane as:

$\mathbf{r}.(-2\mathbf{i}+2\mathbf{j}-\mathbf{k}) = (3\mathbf{i}-3\mathbf{j}-2\mathbf{k}).(-2\mathbf{i}+2\mathbf{j}-\mathbf{k}) = -6-6+2 = -10$

So $2x-2y+z = 10$ is a Cartesian equation of the plane.

d $\overrightarrow{A_1P} = (3\mathbf{i}-\mathbf{j}+2\mathbf{k}) - (3\mathbf{i}-3\mathbf{j}-2\mathbf{k}) = 2\mathbf{j}+4\mathbf{k} = 2\mathbf{b}_1$

$\overrightarrow{A_2P} = (3\mathbf{i}-\mathbf{j}+2\mathbf{k}) - (8\mathbf{i}+3\mathbf{j}) = (-5\mathbf{i}-4\mathbf{j}+2\mathbf{k}) = -\mathbf{b}_2$

Area of triangle $PA_1A_2 = \frac{1}{2}\left|\overrightarrow{A_1P}\times\overrightarrow{A_2P}\right| = \frac{1}{2}\left|2\mathbf{b}_1\times -\mathbf{b}_2\right|$

$= \left|\mathbf{b}_1\times\mathbf{b}_2\right| = \left|-10\mathbf{i}+10\mathbf{j}-5\mathbf{k}\right|$ from part **b**

$= \sqrt{(-10)^2+(10)^2+(-5)^2}$

$= \sqrt{225} = 15$

6 a $\overrightarrow{AB} = (\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) - (-\mathbf{j} + 2\mathbf{k}) = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$\overrightarrow{CD} = (\mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) = -2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$

$$\mathbf{p} = \overrightarrow{AB} \times \overrightarrow{CD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ -2 & 3 & -5 \end{vmatrix} = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

b $\overrightarrow{AC} = (2\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}) - (-\mathbf{j} + 2\mathbf{k}) = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$

$\overrightarrow{AC}.\mathbf{p} = (2\mathbf{i} - \mathbf{j} + 5\mathbf{k}).(\mathbf{i} - \mathbf{j} - \mathbf{k}) = 2 + 1 - 5 = -2$

c The line containing AB has equation $\mathbf{r} = -\mathbf{j} + 2\mathbf{k} + \lambda\overrightarrow{AB}$

The line containing CD has equation $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k} + \mu\overrightarrow{CD}$

So the shortest distance between the lines containing AB and the line containing CD is

$$\left|\frac{(-\mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} - \mathbf{j} + 5\mathbf{k}).\overrightarrow{AB} \times \overrightarrow{CD}}{|\overrightarrow{AB} \times \overrightarrow{CD}|}\right| = \left|\frac{\overrightarrow{AC}.\mathbf{p}}{|\mathbf{p}|}\right| = \frac{2}{\sqrt{1^2 + (-1)^2 + (-1)^2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

7 a $\mathbf{b} = \overrightarrow{OM} = -4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

Let $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then as $\mathbf{a}$ represents a point on the line

$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \times (-4\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 5\mathbf{i} - 10\mathbf{k}$

$$\Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ -4 & 1 & -2 \end{vmatrix} = 5\mathbf{i} - 10\mathbf{k}$$

$\Rightarrow (-2y - z)\mathbf{i} + (2x - 4z)\mathbf{j} + (x + 4y)\mathbf{k} = 5\mathbf{i} - 10\mathbf{k}$

Comparing coefficients gives

$-2y - z = 5$ **(1)**

$2x - 4z = 0$ **(2)**

$x + 4y = -10$ **(3)**

Any value chosen for x will give a point on the line. Let $x = 2$ say

Then from equation **(3)**: $4y = -12 \Rightarrow y = -3$

And from equation **(2)**: $4 - 4z = 0 \Rightarrow z = 1$

Therefore $(2, -3, 1)$ is a point on the line.

So the equation of line may be written as $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + t(-4\imath + \mathbf{j} - 2\mathbf{k})$

b It has already been shown in part **a** that (2, –3, 1) lies on the line l with vector equation $\mathbf{r} \times \overrightarrow{OM} = 5\mathbf{i} - 10\mathbf{k}$, so $\overrightarrow{ON} \times \overrightarrow{OM} = 5\mathbf{i} - 10\mathbf{k}$

$$\text{Area of triangle } OMN = \frac{1}{2}\left|\overrightarrow{OM} \times \overrightarrow{ON}\right| = \frac{1}{2}\left|\overrightarrow{ON} \times \overrightarrow{OM}\right| = \frac{1}{2}|5\mathbf{i} - 10\mathbf{k}|$$

$$= \frac{1}{2}\sqrt{5^2 + (-10)^2} = \frac{\sqrt{125}}{2} = \frac{5\sqrt{5}}{2} = 5.59 \quad (3 \text{ s.f.})$$

8 a $\overrightarrow{AB} = (3\mathbf{i}+\mathbf{j}+4\mathbf{k})-(\mathbf{i}+3\mathbf{j}+3\mathbf{k}) = 2\mathbf{i}-2\mathbf{j}+\mathbf{k}$

$\overrightarrow{AC} = (2\mathbf{i}+4\mathbf{j}+\mathbf{k})-(\mathbf{i}+3\mathbf{j}+3\mathbf{k}) = \mathbf{i}+\mathbf{j}-2\mathbf{k}$

A vector normal to the plane ABC is the direction $\overrightarrow{AB}\times\overrightarrow{AC}$.

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 3\mathbf{i}+5\mathbf{j}+4\mathbf{k}$$

A unit vector normal to the plane is $\dfrac{1}{\sqrt{3^2+5^2+4^2}}(3\mathbf{i}+5\mathbf{j}+4\mathbf{k}) = \dfrac{1}{\sqrt{50}}(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})$

b Using $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$, with $\mathbf{n} = 3\mathbf{i}+5\mathbf{j}+4\mathbf{k}$ and $\mathbf{a} = \mathbf{i}+3\mathbf{j}+3\mathbf{k}$ (note $\mathbf{a}$ can be the position vector of any point on the plane), this gives a vector equation of the plane as:

$\mathbf{r}.(3\mathbf{i}+5\mathbf{j}+4\mathbf{k}) = (\mathbf{i}+3\mathbf{j}+3\mathbf{k}).(3\mathbf{i}+5\mathbf{j}+4\mathbf{k}) = 3+15+12 = 30$

So $3x+5y+4z = 30$ is a Cartesian equation of the plane.

c The perpendicular distance from the origin to a plane with equation $\mathbf{r}.\mathbf{n} = k$ where $\mathbf{n}$ is a unit vector perpendicular to the plane is k.

So from part **b**, the vector equation of the plane is $\mathbf{r}.\dfrac{1}{\sqrt{50}}(3\mathbf{i}+5\mathbf{j}+4\mathbf{k}) = \dfrac{30}{\sqrt{50}}$

So the perpendicular distance from the origin to the plane $= \dfrac{30}{\sqrt{50}} = \dfrac{30\sqrt{50}}{50} = 3\sqrt{2}$

9 a Two non-parallel lines in the plane with vector equation $\mathbf{r} = \mathbf{i}+s\mathbf{j}+t(\mathbf{i}-\mathbf{k})$ are $\mathbf{j}$ and $\mathbf{i}-\mathbf{k}$

So a normal to the plane is $\mathbf{j}\times\mathbf{i}-\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -\mathbf{i}-\mathbf{k}$

As $\mathbf{i}+\mathbf{k}$ is parallel to $-\mathbf{i}-\mathbf{k}$, it must be is perpendicular to the plane.

b From part **b**, $\mathbf{n} = \dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k})$ is a unit vector perpendicular to the plane.

Using $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$, with $\mathbf{a} = \mathbf{i}$, this gives a vector equation of the plane as

$\mathbf{r}.\dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k}) = (\mathbf{i}).\dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k}) = \dfrac{1}{\sqrt{2}}$

So as $\dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k})$ is a unit vector,

the perpendicular distance from the origin to the plane $= \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2} = 0.707$ (3 s.f.)

c As $\mathbf{r}.\dfrac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k}) = \dfrac{1}{\sqrt{2}}$ is a vector equation of the plane

A Cartesian equation of the plane is $\dfrac{1}{\sqrt{2}}(x+z) = \dfrac{1}{\sqrt{2}}$, which simplifies to $x+z=1$

10 a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (5\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 4\mathbf{i} - 3\mathbf{j}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}) - (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

A perpendicular vector to the plane is in direction $\overrightarrow{AB} \times \overrightarrow{AC}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -3 & 0 \\ 2 & 1 & 5 \end{vmatrix} = -15\mathbf{i} - 20\mathbf{j} + 10\mathbf{k}$$

b The equation of the plane containing *A*, *B* and *C* is

Using $\mathbf{r.n} = \mathbf{a.n}$, with $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, this gives a vector equation of the plane as

$\mathbf{r}.(-15\mathbf{i} - 20\mathbf{j} + 10\mathbf{k}) = (\mathbf{i} + \mathbf{j} + \mathbf{k}).(-15\mathbf{i} - 20\mathbf{j} + 10\mathbf{k}) = -15 - 20 + 10 = -25$

So a Cartesian equation of the plane is $-15x - 20y + 10z = -25$, which simplifies to $3x + 4y - 2z = 5$

c $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}) - (\mathbf{i} + \mathrm{j} + \mathbf{k}) = 4\mathbf{j} + 5\mathbf{k}$

Volume of tetrahedron $ABCD = \frac{1}{6}\left|\overrightarrow{AD}.(\overrightarrow{AB} \times \overrightarrow{AC})\right|$

$$= \frac{1}{6}\left|(4\mathbf{j} + 5\mathbf{k}).(-15\mathbf{i} - 20\mathbf{j} + 10\mathbf{k})\right| = \frac{1}{6}\left|(-80 + 50)\right| = \frac{30}{6} = 5$$

11 a $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (2\mathbf{i} - 3\mathbf{j}) - (3\mathbf{i} - 5\mathbf{j} - \mathbf{k}) = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (2\mathbf{i} - 3\mathbf{j}) - (-\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}) = 3\mathbf{i} - 8\mathbf{j} - 7\mathbf{k}$

$$\overrightarrow{AC} \times \overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 3 & -8 & -7 \end{vmatrix} = -6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$$

b $\overrightarrow{AB} \times \overrightarrow{AC}$ is a normal to the plane Π and $3\mathbf{i} - 5\mathbf{j} - \mathbf{k}$ is a point on the plane

So an equation of the plane is

$\mathbf{r}.(-6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = (3\mathbf{i} - 5\mathbf{j} - \mathbf{k}).(-6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = -18 + 20 - 2 = 0$

This simplifies to $\mathbf{r}.(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 0$

c As $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is a normal to the plane, the perpendicular from the point (2, 3, –2) to the plane has the equation

$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

Using the result from part **b**, this meets the plane when

$\left((2 + 3\lambda)\mathbf{i} + (3 + 2\lambda)\mathbf{j} + (-2 - \lambda)\mathbf{k}\right).(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 0$

$\Rightarrow 3(2 + 3\lambda) + 2(3 + 2\lambda) - 1(-2 - \lambda) = 0$

$\Rightarrow 14\lambda + 14 = 0$

$\Rightarrow \lambda = -1$

Substitute $\lambda = -1$ into the equation of the line gives

$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + (-1)(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -\mathbf{i} + \mathbf{j} - \mathbf{k}$

So the perpendicular from (2, 3, –2) meets the plane at (–1, 1, –1)

12 a $\mathbf{p}\times\mathbf{q}=(3\mathbf{i}-\mathbf{j}+2\mathbf{k})\times(2\mathbf{i}+\mathbf{j}-\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -1 & 2\\ 2 & 1 & -1\end{vmatrix}=-\mathbf{i}+7\mathbf{j}+5\mathbf{k}$

b $\mathbf{p}\times\mathbf{q}$ is a normal to the plane and the point with position vector $3\mathbf{i}-\mathbf{j}+2\mathbf{k}$ is on the plane, so an equation of the plane is

$\mathbf{r}.(-\mathbf{i}+7\mathbf{j}+5\mathbf{k})=(3\mathbf{i}-\mathbf{j}+2\mathbf{k}).(-\mathbf{i}+7\mathbf{j}+5\mathbf{k})=-3-7+10=0$

So a Cartesian equation for the plane is $-x+7y+5z=0$

c $(\mathbf{r}-\mathbf{p})\times\mathbf{q}=0$ is one form of the vector equation of a line passing through the point with position vector **p** and parallel to the vector **q**. So the equation can also be written as

$\mathbf{r}=\mathbf{pq}+\lambda\mathbf{q}$, i.e. $\mathbf{r}=3\mathbf{i}-\mathbf{j}+2\mathbf{k}+\lambda(2\mathbf{i}+\mathbf{j}-\mathbf{k})$

This meets the plane $\mathbf{r}.(\mathbf{i}+\mathbf{j}+\mathbf{k})=2$ when

$\big((3+2\lambda)+(-1+\lambda)+(2-\lambda)\big).(\mathbf{i}+\mathbf{j}+\mathbf{k})=2$

$\Rightarrow(3+2\lambda)+(-1+\lambda)+(2-\lambda)=2\Rightarrow 2\lambda+4=2\Rightarrow\lambda=-1$

Substitute $\lambda=-1$ into the equation of the line gives

$\mathbf{r}=3\mathbf{i}-\mathbf{j}+2\mathbf{k}+(-1)(2\mathbf{i}+\mathbf{j}-\mathbf{k})=\mathbf{i}-2\mathbf{j}+3\mathbf{k}$

So the coordinates of point T are (1, –2, 3)

13 a Let the respective normal to each plane be $\mathbf{n}_1$ and $\mathbf{n}_2$, then

$\mathbf{n}_1=2\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $\mathbf{n}_2=\mathbf{i}-2\mathbf{j}$

Let the acute angle between the two planes be θ, then θ is also the angle between the respective normal to each plane, so

$$\cos\theta=\left|\frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right|=\frac{|2\times1-2\times2|}{\sqrt{2^2+2^2+(-1)^2}\sqrt{1^2+(-2)^2}}=\frac{2}{3\sqrt{5}}=\frac{2\sqrt{5}}{15}$$

$\Rightarrow\theta=72.7°=73°$ (to the nearest degree)

13 b The direction of the line of intersection is perpendicular to the normal of each plane.

Hence the direction is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 1 & -2 & 0 \end{vmatrix} = -2\mathbf{i} - \mathbf{j} - 6\mathbf{k}$

Any scalar multiple of this vector is also in the direction of the line of intersection, so simplify by multiplying by -1 to get $2\mathbf{i} + \mathbf{j} + 6\mathbf{k}$

Find a point on the line by setting $y = 0$ and solving the Cartesian equations of the two planes.

$2x + 2y - z = 9$ **(1)**

$x - 2y = 7$ **(2)**

Substituting for y in equation **(2)** gives: $x = 7$

Substituting for x and y in equation **(1)** gives: $2 \times 7 - z = 9 \Rightarrow z = 5$

So $7\mathbf{i} + 5\mathbf{k}$ is the position vector of a point on the line of intersection

A line passing through a point with position vector **a** and parallel to vector **b** has the vector equation $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$, so an equation of the line of intersection is

$$\mathbf{r} \times (2\mathbf{i} + \mathbf{j} + 6\mathbf{k}) = (7\mathbf{i} + 5\mathbf{k}) \times (2\mathbf{i} + \mathbf{j} + 6\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 0 & 5 \\ 2 & 1 & 6 \end{vmatrix} = -5\mathbf{i} - 32\mathbf{j} + 7\mathbf{k}$$

So the equation is $\mathbf{r} \times (2\mathbf{i} + \mathbf{j} + 6\mathbf{k}) = -5\mathbf{i} - 32\mathbf{j} + 7\mathbf{k}$

14 a The normal to the plane Π is in the direction

$$(4\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \times (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 1 & 2 \\ 3 & 2 & -1 \end{vmatrix} = -5\mathbf{i} + 10\mathbf{j} + 5\mathbf{k}$$

The line L is in the direction $(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$

Finding the scalar product of the direction of the normal to the plane and the direction of the line

$(-5\mathbf{i} + 10\mathbf{j} + 5\mathbf{k}).(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = -10 + 30 - 20 = 0$

This means that the line L is perpendicular to the normal to the plane, so the line L is parallel to the plane Π.

14 b The line L passes through point (2, 1, –3)

The perpendicular to plane Π through the point (2, 1, –3) has a vector equation

$\mathbf{r} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} + \lambda(-5\mathbf{i} + 10\mathbf{j} + 5\mathbf{k})$

As the normal to the plane is $-5\mathbf{i} + 10\mathbf{j} + 5\mathbf{k}$ and $\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ is the position vector of a point on the plane, the equation of the plane may be written as

$\mathbf{r}.(-5\mathbf{i} + 10\mathbf{j} + 5\mathbf{k}) = (\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}).(-5\mathbf{i} + 10\mathbf{j} + 5\mathbf{k}) = -5 + 30 + 20 = 45$

So the perpendicular to the plane Π from (2, 1, –3) meets the plane when

$\big((2 - 5\lambda)\mathbf{i} + (1 + 10\lambda)\mathbf{j} + (-3 + 5\lambda)\mathbf{k}\big).(-5 + 10\mathbf{j} + 5\mathbf{k}) = 45$

$\Rightarrow -10 + 25\lambda + 10 + 100\lambda - 15 + 25\lambda = 45$

$\Rightarrow 150\lambda = 60 \Rightarrow \lambda = \dfrac{2}{5}$

Substituting $\lambda = \dfrac{2}{5}$ into the equation of the perpendicular to plane Π through the point (2, 1, –3) gives $\mathbf{r} = 5\mathbf{j} - \mathbf{k}$, so the perpendicular to Π from (2, 1, –3) meets the plane at (0, 5, –1). As the line is parallel to the plane, the shortest distance from L to Π is the distance between these points, i.e.

$$\sqrt{(2-0)^2 + (1-5)^2 + (-3-(-1))^2} = \sqrt{4+16+4} = \sqrt{24} = 2\sqrt{6} = 4.90 \quad \text{(3 s.f.)}$$

Alternatively, note that as L is parallel to the plane Π, the shortest distance between L and the plane will also be the shortest distance between L and any line L_1 on the plane that is non-parallel with L. These two lines are skew.

Write the equation of L as $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

And L_1 as $\mathbf{r} = \mathbf{c} + s\mathbf{d}$, where $\mathbf{c} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, a point on the plane, and $\mathbf{d} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, a direction on the plane

$$\mathbf{b} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ 4 & 1 & 2 \end{vmatrix} = 10\mathbf{i} - 20\mathbf{j} - 10\mathbf{k}$$

Using the result for the shortest distance between two skew lines

$$\text{Shortest distance} = \left|\frac{(\mathbf{a}-\mathbf{c}).(\mathbf{b}\times\mathbf{d})}{|(\mathbf{b}\times\mathbf{d})|}\right| = \left|\frac{(\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}).(10\mathbf{i} - 20\mathbf{j} - 10\mathbf{k})}{\sqrt{10^2 + (-20)^2 + (-10)^2}}\right|$$

$$= \frac{10 + 40 + 70}{\sqrt{600}} = \frac{120}{10\sqrt{6}} = \frac{12}{\sqrt{6}} = 2\sqrt{6} = 4.90 \quad \text{(3 s.f.)}$$

15 a A normal to the plane is $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ so the line l is parallel to this vector and it passes through the point with position vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, hence a vector equation of the line is

$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

b From the vector equation, the coordinates of a point on l are $(1 + 2\lambda, 2 + \lambda, 1 + 3\lambda)$

So the line l meets the plane Π when

$2(1 + 2\lambda) + (2 + \lambda) + 3(1 + 3\lambda) = 21$

$\Rightarrow 14\lambda + 7 = 21 \Rightarrow \lambda = 1$

Substitute $\lambda = 1$ into the equation of the line l gives $\mathbf{r} = 3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

So M has coordinates (3, 3, 4)

15 c $\overrightarrow{OP} \times \overrightarrow{OM} = (\mathbf{i}+2\mathbf{j}+\mathbf{k}) \times (3\mathbf{i}+3\mathbf{j}+4\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 3 & 3 & 4 \end{vmatrix} = 5\mathbf{i}-\mathbf{j}-3\mathbf{k}$

d Let θ be the acute angle between the vectors $\overrightarrow{OP}$ and $\overrightarrow{OM}$

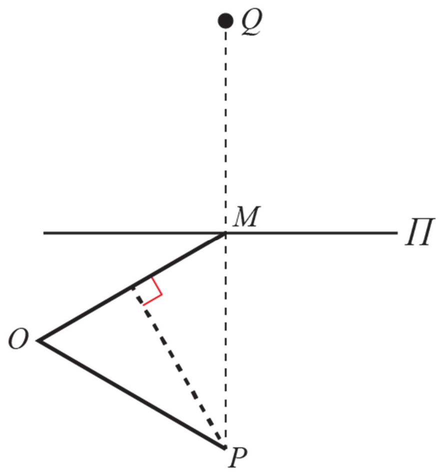
Then, by simple geometry, the distance d from P to the line OM is $|\overrightarrow{OP}|\sin\theta$

From the definition of the vector product $\sin\theta = \dfrac{|\overrightarrow{OP} \times \overrightarrow{OM}|}{|\overrightarrow{OP}||\overrightarrow{OM}|}$

$$\text{So } d = |\overrightarrow{OP}|\sin\theta = \frac{|\overrightarrow{OP}||\overrightarrow{OP} \times \overrightarrow{OM}|}{|\overrightarrow{OP}||\overrightarrow{OM}|} = \frac{|\overrightarrow{OP} \times \overrightarrow{OM}|}{|\overrightarrow{OM}|}$$

$$= \frac{\sqrt{5^2+(-1)^2+(-3)^2}}{\sqrt{3^2+3^2+4^2}} = \frac{\sqrt{35}}{\sqrt{34}}$$

e This sketch shows the problem



$\overrightarrow{PM} = (3\mathbf{i}+3\mathbf{j}+4\mathbf{k}) - (\mathbf{i}+2\mathbf{j}+\mathbf{k}) = 2\mathbf{i}+\mathbf{j}+3\mathbf{k}$

Therefore $\overrightarrow{MQ} = 2\mathbf{i}+\mathbf{j}+3\mathbf{k}$

And $\overrightarrow{OQ} = \overrightarrow{OM} + \overrightarrow{MQ} = (3\mathbf{i}+3\mathbf{j}+4\mathbf{k}) + (2\mathbf{i}+\mathbf{j}+3\mathbf{k}) = 5\mathbf{i}+4\mathbf{j}+7\mathbf{k}$

So Q has coordinates (5, 4, 7)

16 a $\overrightarrow{BC} = (2\mathbf{i}+3\mathbf{j}+3\mathbf{k})-(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = \mathbf{i}+\mathbf{j}$

$\overrightarrow{BD} = (3\mathbf{i}+2\mathbf{j}+4\mathbf{k})-(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = 2\mathbf{i}+\mathbf{k}$

So $\overrightarrow{BC}\times\overrightarrow{BD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = \mathbf{i}-\mathbf{j}-2\mathbf{k}$ which is normal to the plane BCD

Using $\mathbf{r.n} = \mathbf{a.n}$, with $\mathbf{a} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$, this gives a vector equation of the plane BCD as

$\mathbf{r}.(\mathbf{i}-\mathbf{j}-2\mathbf{k}) = (\mathbf{i}+2\mathbf{j}+3\mathbf{k}).(\mathbf{i}-\mathbf{j}-2\mathbf{k}) = 1-2-6 = -7$

This may be written in Cartesian form as $x-y-2z+7=0$

b Let α be the angle between BC and the plane $x+2y+3z=4$ and θ be the acute angle between BC and the normal to this plane, which is $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$.

Then $\alpha = 90-\theta \Rightarrow \sin\alpha = \cos\theta$

So $\sin\alpha = \cos\theta = \dfrac{|(\mathbf{i}+\mathbf{j}).(\mathbf{i}+2\mathbf{j}+3\mathbf{k})|}{\sqrt{1^2+1^2}\sqrt{1^2+2^2+3^2}} = \dfrac{3}{\sqrt{2}\sqrt{14}} = 0.567$ (3 s.f.)

c Let A have coordinates (x, y, z)

Then $\overrightarrow{AC} = (2-x)\mathbf{i}+(3-y)\mathbf{j}+(3-z)\mathbf{k}$ and $\overrightarrow{AD} = (3-x)\mathbf{i}+(2-y)\mathbf{j}+(4-z)\mathbf{k}$

As AC is perpendicular to BD, $\overrightarrow{AC}.\overrightarrow{BD} = 0$

So $2(2-x)+(3-z)=0$

$\Rightarrow 2x+z=7$ **(1)**

As AD is perpendicular to BC, $\overrightarrow{AD}.\overrightarrow{BC} = 0$

So $(3-x)+(2-y)=0$

$\Rightarrow x+y=5$ **(2)**

As $AB = \sqrt{26}$

$(x-1)^2+(y-2)^2+(z-3)^2 = 26$ **(3)**

Substituting $z = 7-2x$ and $y = 5-x$ from equation **(1)** and **(2)** into equation **(3)** gives

$(x-1)^2+(3-x)^2+(4-2x)^2 = 26$

$x^2-2x+1+9-6x+x^2+16-16x+4x^2 = 26$

$6x^2-24x = 0$

$x(x-4) = 0$

$\Rightarrow x = 0$ or 4

When $x=0$, $y=5$ and $z=7$

When $x=4$, $y=1$ and $z=-1$

The two possible positions are (0, 5, 7) and (4, 1, –1)

17 a $\overrightarrow{AB} = (4\mathbf{i}+2\mathbf{j}-3\mathbf{k})-(-2\mathbf{i}+\mathbf{j}+5\mathbf{k}) = 6\mathbf{i}+\mathbf{j}-8\mathbf{k}$

Hence the direction ratios are $6:1:-8$

b The direction cosines are given by

$$l = \frac{6}{\sqrt{36+1+64}} = \frac{6}{\sqrt{101}}$$

$$m = \frac{1}{\sqrt{101}}$$

$$n = -\frac{8}{\sqrt{101}}$$

c As the point A lies on the line, the Cartesian equation of the line is

$$\frac{x+2}{l} = \frac{y-1}{m} = \frac{z-5}{n}, \text{ i.e. } \frac{x+2}{\left(\frac{6}{\sqrt{101}}\right)} = \frac{y-1}{\left(\frac{1}{\sqrt{101}}\right)} = \frac{z-5}{\left(-\frac{8}{\sqrt{101}}\right)}$$

18 The direction cosines for the line are

$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma$

Using the identity $l^2 + m^2 + n^2 = 1$ gives

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Using the trigonometric identity $\cos^2\theta = 1 - \sin^2\theta$ gives

$3 - \sin^2\alpha - \sin^2\beta - \sin^2\gamma = 1$

Hence $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

19 L_1 has direction vector $\mathbf{r}_1 = \lambda\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}$

Similarly L_2 has direction vector $\mathbf{r}_2 = \mu\begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix}$

For some real numbers λ and μ

Since the lines are parallel, $\mathbf{r}_1 = t\mathbf{r}_2$ for some real number t hence

$$\lambda l_1 = t\mu l_2 \Rightarrow \frac{l_1}{l_2} = \frac{t\mu}{\lambda}$$

$$\lambda m_1 = t\mu m_2 \Rightarrow \frac{m_1}{m_2} = \frac{t\mu}{\lambda}$$

$$\lambda n_1 = t\mu n_2 \Rightarrow \frac{n_1}{n_2} = \frac{t\mu}{\lambda}$$

So $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

20 a The direction cosines for W_1 are

$$l = \cos 45^\circ = \frac{1}{\sqrt{2}}, \quad m = \cos 60^\circ = \frac{1}{2}, \quad n = \cos 60^\circ = \frac{1}{2}$$

Hence an equation for W_1 is $\mathbf{r} = \mu\begin{pmatrix}\sqrt{2}\\1\\1\end{pmatrix}$

The lines intersect if

$$\mathbf{r} = \mu\begin{pmatrix}\sqrt{2}\\1\\1\end{pmatrix} = \begin{pmatrix}\frac{8+3\sqrt{2}}{4}\\0\\\frac{5\sqrt{2}}{4}\end{pmatrix} + \lambda\begin{pmatrix}3\\-4\\1\end{pmatrix}$$

This gives

$$\sqrt{2}\mu = \frac{8+3\sqrt{2}}{4} + 3\lambda \qquad \textbf{(1)}$$

$$\mu = -4\lambda \qquad \textbf{(2)}$$

$$\mu = \frac{5\sqrt{2}}{4} + \lambda \qquad \textbf{(3)}$$

Substituting equation **(2)** in equation **(3)** gives

$$-4\lambda = \frac{5\sqrt{2}}{4} + \lambda \Rightarrow \lambda = -\frac{\sqrt{2}}{4}$$

Substituting for λ in equation **(2)** gives $\mu = \sqrt{2}$

Equation **(1)** holds for these values of λ and μ as $\sqrt{2}\times\sqrt{2} = 2 = \frac{8+3\sqrt{2}}{4} - \frac{3\sqrt{2}}{4}$

Hence there is a solution, so the lines intersect at point A and substituting for $\mu = 2$ in the equation for W_1 gives the coordinates of A as $(2, \sqrt{2}, \sqrt{2})$

b As the pylon lies in the xy-plane and is perpendicular to that plane $\overrightarrow{AB} = -\sqrt{2}\mathbf{k}$ and $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = 2\mathbf{i} - \sqrt{2}\mathbf{j} + \sqrt{2}\mathbf{k} - \sqrt{2}\mathbf{k} = 2\mathbf{i} - \sqrt{2}\mathbf{j}$

Distance from origin to point $B = \left|\overrightarrow{OB}\right| = \sqrt{2^2 + \left(\sqrt{2}\right)^2} = \sqrt{6}$

Converting model units to metres, the distance $= 10\sqrt{6} = 24.5$ m (3 s.f.)

c In reality the wires won't be perfectly straight, as there will be some sag.

21 Find the direction vectors of the new plane by transforming the direction vectors for the original plane giving

$$\begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & -1 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \\ 4 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & -1 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \\ 8 \end{pmatrix}$$

Hence the normal to the transformed plane is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -8 & 4 \\ 2 & -8 & 8 \end{vmatrix} = -32\mathbf{i} - 40\mathbf{j} - 32\mathbf{k}$$

Simplify by dividing by –8, so $4\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$ is a normal to Π_2

Find a point on Π_2 by transforming a point on Π_1

$$\begin{pmatrix} 1 & 0 & 3 \\ 1 & -2 & -1 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -3 \end{pmatrix}$$

So $-2\mathbf{i} - 3\mathbf{k}$ is a point on Π_2

Using $\mathbf{r.n} = \mathbf{a.n}$, with $\mathbf{n} = 4\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$ and $\mathbf{a} = -2\mathbf{i} - 3\mathbf{k}$, this gives a vector equation of the plane as:

$$\mathbf{r}.(4\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}) = (-2\mathbf{i} - 3\mathbf{k}).(4\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}) = -8 - 12 = -20$$

This can be written $\mathbf{r}.\begin{pmatrix} 4 \\ 5 \\ 4 \end{pmatrix} = -20$

Challenge

Two direction vectors in the plane given by $\mathbf{r}_1 = p\mathbf{i} - r\mathbf{k}$ and $\mathbf{r}_2 = q\mathbf{j} - r\mathbf{k}$
Hence a normal to the plane is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ p & 0 & -r \\ 0 & q & -r \end{vmatrix} = qr\mathbf{i} + pr\mathbf{j} + pq\mathbf{k}$$

Using $\mathbf{r.n} = \mathbf{a.n}$, with $\mathbf{n} = qr\mathbf{i} + pr\mathbf{j} + pq\mathbf{k}$ and $\mathbf{a} = p\mathbf{i}$, a point on the plane, this gives a vector equation of the plane as:

$$\mathbf{r}.qr\mathbf{i} + pr\mathbf{j} + pq\mathbf{k} = p\mathbf{i}.(qr\mathbf{i} + pr\mathbf{j} + pq\mathbf{k}) = pqr$$

If d is the length of the perpendicular from the origin to the plane then $\mathbf{r}.\frac{1}{|\mathbf{n}|}\mathbf{n} = d$

So $d = \dfrac{pqr}{\sqrt{q^2r^2 + p^2r^2 + p^2q^2}}$

$$\Rightarrow d^2 = \frac{p^2q^2r^2}{q^2r^2 + p^2r^2 + p^2q^2}$$

$$\Rightarrow \frac{1}{d^2} = \frac{q^2r^2 + p^2r^2 + p^2q^2}{p^2q^2r^2} = \frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}$$