Vectors 1D

1 **a** The equation of the line is $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0\Rightarrow\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$. This gives:

$$\mathbf{r}\times(3\mathbf{i}+\mathbf{j}-2\mathbf{k})=(2\mathbf{i}+\mathbf{j}+2\mathbf{k})\times(3\mathbf{i}+\mathbf{j}-2\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&2\\3&1&-2\end{vmatrix}$$

$$\Rightarrow\mathbf{r}\times(3\mathbf{i}+\mathbf{j}-2\mathbf{k})=-4\mathbf{i}+10\mathbf{j}-\mathbf{k}$$

b The equation of the line is $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0\Rightarrow\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$. This gives:

$$\mathbf{r}\times(\mathbf{i}+\mathbf{j}+5\mathbf{k})=(2\mathbf{i}-3\mathbf{k})\times(\mathbf{i}+\mathbf{j}+5\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&0&-3\\1&1&5\end{vmatrix}$$

$$\Rightarrow\mathbf{r}\times(\mathbf{i}+\mathbf{j}+5\mathbf{k})=3\mathbf{i}-13\mathbf{j}+2\mathbf{k}$$

c The equation of the line is $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0\Rightarrow\mathbf{r}\times\mathbf{b}=\mathbf{a}\times\mathbf{b}$. This gives:

$$\mathbf{r}\times(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})=(4\mathbf{i}-2\mathbf{j}+\mathbf{k})\times(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\4&-2&1\\-1&-2&3\end{vmatrix}$$

$$\Rightarrow\mathbf{r}\times(-\mathbf{i}-2\mathbf{j}+3\mathbf{k})=-4\mathbf{i}-13\mathbf{j}-10\mathbf{k}$$

2 Let $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

Then the Cartesian form of the equation of the line that passes through a point with position vector **a** and is parallel to the vector **b** is $\dfrac{x-a_1}{b_1}=\dfrac{y-a_2}{b_2}=\dfrac{z-a_3}{b_3}=\lambda$ (from Core Pure Book 1, Section 9.1)

a $\dfrac{x-2}{3}=\dfrac{y-1}{1}=\dfrac{z-2}{-2}=\lambda$

b $\dfrac{x-2}{1}=\dfrac{y}{1}=\dfrac{z+3}{5}=\lambda$

c $\dfrac{x-4}{-1}=\dfrac{y+2}{-2}=\dfrac{z-1}{3}=\lambda$

3 **a** The line is in the direction $\begin{pmatrix}6\\4\\2\end{pmatrix}-\begin{pmatrix}1\\3\\5\end{pmatrix}=\begin{pmatrix}5\\1\\-3\end{pmatrix}$

The equation is $\left(\mathbf{r}-\begin{pmatrix}1\\3\\5\end{pmatrix}\right)\times\begin{pmatrix}5\\1\\-3\end{pmatrix}=0$

3 b The line is in the direction $\begin{pmatrix}4\\3\\5\end{pmatrix}-\begin{pmatrix}3\\4\\12\end{pmatrix}=\begin{pmatrix}1\\-1\\-7\end{pmatrix}$

The equation is $\left(\mathbf{r}-\begin{pmatrix}3\\4\\12\end{pmatrix}\right)\times\begin{pmatrix}1\\-1\\-7\end{pmatrix}=0$

c The line is in the direction $\begin{pmatrix}3\\7\\11\end{pmatrix}-\begin{pmatrix}-2\\2\\6\end{pmatrix}=\begin{pmatrix}5\\5\\5\end{pmatrix}$

The equation is $\left(\mathbf{r}-\begin{pmatrix}-2\\2\\6\end{pmatrix}\right)\times\begin{pmatrix}5\\5\\5\end{pmatrix}=0$

d The line is in the direction $\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}4\\2\\-4\end{pmatrix}=\begin{pmatrix}-3\\-1\\5\end{pmatrix}$

The equation is $\left(\mathbf{r}-\begin{pmatrix}1\\1\\1\end{pmatrix}\right)\times\begin{pmatrix}-3\\-1\\5\end{pmatrix}=0$

In each part of question **3** one solution is illustrated, but there are alternatives. Either given point may be used for **a** in the equation $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$ and any multiple of the direction vector may be used as **b**.

4 The solutions for question **3** are in the form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$, where **a** is the position vector of a point on the line and **b** is vector parallel to the line. Use the standard formula for the Cartesian form of the equation of the line (see solution to question **2**). As with question **3**, there are alternative solutions.

a $\frac{x-1}{5}=\frac{y-3}{1}=\frac{z-5}{-3}=\lambda$

b $\frac{x-3}{1}=\frac{y-4}{-1}=\frac{z-12}{-7}=\lambda$

c $\frac{x+2}{5}=\frac{y-2}{5}=\frac{z-6}{5}=\lambda$

Or $x+2=y-2=z-6=\mu$ as $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is also in the direction of the line.

4 d $\dfrac{x-1}{-3}=\dfrac{y-1}{-1}=\dfrac{z-1}{5}=\lambda$

Or $\dfrac{x-4}{3}=\dfrac{y-2}{1}=\dfrac{z+4}{-5}=\lambda$ as (4, 2, –4) is a point on the line and $3\mathbf{i}+\mathbf{j}-5\mathbf{k}$ is also in the direction of the line.

5 A straight line with the equation $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ passes through the point with position vector **a** and is parallel to the vector **b**. The equation of the line can be written $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$.

a $(\mathbf{r}-(\mathbf{i}+\mathbf{j}-2\mathbf{k}))\times(2\mathbf{i}-\mathbf{k})=0$

b $(\mathbf{r}-(\mathbf{i}+4\mathbf{j}))\times(3\mathbf{i}+\mathbf{j}-5\mathbf{k})=0$

c $(\mathbf{r}-(3\mathbf{i}+4\mathbf{j}-4\mathbf{k}))\times(2\mathbf{i}-2\mathbf{j}-3\mathbf{k})=0$

6 i $\dfrac{x-3}{2}=\dfrac{y+1}{5}=\dfrac{2z-3}{3}=\lambda$ can be written as $\dfrac{x-3}{2}=\dfrac{y+1}{5}=\dfrac{z-\frac{3}{2}}{\frac{3}{2}}=\lambda$

The direction of the line is parallel $2\mathbf{i}+5\mathbf{j}+\dfrac{3}{2}\mathbf{k}$

A point on the line has position vector $3\mathbf{i}-\mathbf{j}+\dfrac{3}{2}\mathbf{k}$

Therefore the vector equation of the line can be written as

$$\mathbf{r}\times\left(2\mathbf{i}+5\mathbf{j}+\frac{3}{2}\mathbf{k}\right)=\left(3\mathbf{i}-\mathbf{j}+\frac{3}{2}\mathbf{k}\right)\times\left(2\mathbf{i}+5\mathbf{j}+\frac{3}{2}\mathbf{k}\right)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -1 & \frac{3}{2}\\ 2 & 5 & \frac{3}{2}\end{vmatrix}$$

$$\Rightarrow \mathbf{r}\times\left(2\mathbf{i}+5\mathbf{j}+\frac{3}{2}\mathbf{k}\right)=-9\mathbf{i}-\frac{3}{2}\mathbf{j}+17\mathbf{k}$$

ii $\mathbf{r}=3\mathbf{i}-\mathbf{j}+\dfrac{3}{2}\mathbf{k}+t\left(2\mathbf{i}+5\mathbf{j}+\dfrac{3}{2}\mathbf{k}\right)$

Or $\mathbf{r}=3\mathbf{i}-\mathbf{j}+\dfrac{3}{2}\mathbf{k}+s(4\mathbf{i}+10\mathbf{j}+3\mathbf{k})$

7 As $(p, q, 1)$ lies on the line with equation $\mathbf{r}\times\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}8\\-7\\-3\end{pmatrix}$ then $\begin{pmatrix}p\\q\\1\end{pmatrix}\times\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}8\\-7\\-3\end{pmatrix}$

$$\begin{pmatrix}p\\q\\1\end{pmatrix}\times\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ p & q & 1\\ 2 & 1 & 3\end{vmatrix}==(3q-1)\mathbf{i}-(3p-2)\mathbf{j}+(p-2q)\mathbf{k}=\begin{pmatrix}3q-1\\2-3p\\p-2q\end{pmatrix}$$

So $3q-1=8\Rightarrow q=3$ and $2-3p=-7\Rightarrow p=3$

Solution: $p=3$ and $q=3$

8 The line with equation $\mathbf{r}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$ has direction $\mathbf{i}+\mathbf{j}-\mathbf{k}$, i.e. $\begin{pmatrix}1\\1\\-1\end{pmatrix}$

If the line passes through a point (a_1, a_2, a_3) then $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\1&1&-1\end{vmatrix}=(-a_2-a_3)\mathbf{i}+(a_1+a_3)\mathbf{j}+(a_1-a_2)\mathbf{k}=\begin{pmatrix}-a_2-a_3\\a_1+a_3\\a_1-a_2\end{pmatrix}$$

So $\begin{pmatrix}-a_2-a_3\\a_1+a_3\\a_1-a_2\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$

These equations have an infinite number of solutions so let $a_1=0$, then as $a_1+a_3=2$ and $a_1-a_2=1$ this gives $a_3=2$ and $a_2=-1$, therefore (0, –1, 2) lies on the line.

So the line equation may be written as $\mathbf{r}=-\mathbf{j}+2\mathbf{k}+t(\mathbf{i}+\mathbf{j}-\mathbf{k})$

9 a The direction vector for the line is $(3\mathbf{i}+4\mathbf{j}-5\mathbf{k})-(-3\mathbf{i}+2\mathbf{j}+7\mathbf{k})=6\mathbf{i}+2\mathbf{j}-12\mathbf{k}$

Hence the direction cosines are given by:

$$l=\frac{6}{\sqrt{36+4+144}}=\frac{6}{\sqrt{184}}=\frac{3}{\sqrt{46}}$$

$$m=\frac{2}{\sqrt{184}}=\frac{1}{\sqrt{46}}$$

$$n=\frac{-12}{\sqrt{184}}=\frac{-6}{\sqrt{46}}$$

b A Cartesian equation of the line is $\frac{x-3}{l}=\frac{y-4}{m}=\frac{z+5}{n}$

Substituting for *l*, *m* and *n* and dividing by $\sqrt{46}$, this simplifies to $\frac{x-3}{3}=y-4=\frac{z+5}{-6}$

Alternatively use the fact that the Cartesian form of the equation of the line that passes through a point with position vector **a** and is parallel to the vector **b** is $\frac{x-a_1}{b_1}=\frac{y-a_2}{b_2}=\frac{z-a_3}{b_3}$

The line passes through $3\mathbf{i}+4\mathbf{j}-5\mathbf{k}$ and is parallel to $6\mathbf{i}+2\mathbf{j}-12\mathbf{k}$

So an equation of the line is $\frac{x-3}{6}=\frac{y-4}{2}=\frac{z+5}{-12}$

10 a The direction vector for the x-axis is just $\mathbf{i}$ hence the direction cosines are 1, 0, 0

b The direction vector for the y-axis is just $\mathbf{j}$ hence the direction cosines are 0, 1, 0

c The direction vector for the z-axis is just $\mathbf{k}$ hence the direction cosines are 0, 0, 1

d The direction vector for the line $x = y = z$ is $\mathbf{i}+\mathbf{j}+\mathbf{k}$

So the direction cosines are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

11 a The direction vector of L_1 is $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ so:

$$l_1 = \frac{1}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}} \quad m_1 = \frac{2}{\sqrt{14}} \quad n_1 = \frac{3}{\sqrt{14}}$$

b The direction vector of L_2 is $3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ so:

$$l_2 = \frac{3}{\sqrt{1+4+9}} = \frac{3}{\sqrt{14}} \quad m_2 = \frac{2}{\sqrt{14}} \quad n_2 = \frac{1}{\sqrt{14}}$$

c Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the direction vectors of the two lines.

The angle between the two lines θ satisfies $\cos\theta = \frac{\mathbf{r}_1.\mathbf{r}_2}{|\mathbf{r}_1||\mathbf{r}_2|}$

So $\cos\theta = \frac{(\mathbf{i}+2\mathbf{j}+3\mathbf{k}).(3\mathbf{i}+2\mathbf{j}+\mathbf{k})}{\sqrt{14}\times\sqrt{14}} = \frac{3+4+3}{14} = \frac{10}{14} = \frac{5}{7}$

$$l_1l_2 + m_1m_2 + n_1n_2 = \frac{1}{\sqrt{14}}\times\frac{3}{\sqrt{14}} + \frac{2}{\sqrt{14}}\times\frac{2}{\sqrt{14}} + \frac{3}{\sqrt{14}}\times\frac{1}{\sqrt{14}} = \frac{3+4+3}{14} = \frac{5}{7}$$

So $l_1l_2 + m_1m_2 + n_1n_2 = \cos\theta$

d Given any pair of intersecting lines with direction vectors $\mathbf{s}_1$ and $\mathbf{s}_2$, divide each direction vector by a constant to produce direction vectors $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ such that $|\mathbf{r}_1| = 1$ and $|\mathbf{r}_2| = 1$

So $l_1 = \frac{x_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}} = x_1$ as $|\mathbf{r}_1| = 1,\ x_1^2 + y_1^2 + z_1^2 = 1$

And $m_1 = y_1, n_1 = z_1, l_2 = x_2, m_2 = y_2, n_2 = z_2$

Therefore $\cos\theta = \frac{\mathbf{r}_1.\mathbf{r}_2}{|\mathbf{r}_1||\mathbf{r}_2|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}} = x_1x_2 + y_1y_2 + z_1z_2 = l_1l_2 + m_1m_2 + n_1n_2$

12 Use the formula $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$ (see question **11d**)

This gives $\cos\theta = \frac{-3+2+3}{\sqrt{11\times14}} = \frac{2}{\sqrt{154}}$

$\theta = \cos^{-1}\left(\frac{2}{\sqrt{154}}\right) = 1.41$ (3 s.f.)

13 The direction cosines for this line are by definition: $l = \cos\alpha \quad m = \cos\beta \quad n = \cos\gamma$

As $l^2 + m^2 + n^2 = 1$, this gives $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Using the trigonometric identity $\cos 2x \equiv 2\cos^2 x - 1$ this gives:

$$\begin{aligned}\cos 2\alpha + \cos 2\beta + \cos 2\gamma &= 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1\\ &= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3\\ &= -1\end{aligned}$$

14 The direction cosines are: $l = \frac{2}{\sqrt{4+9+16}} = \frac{2}{\sqrt{29}} \quad m = \frac{3}{\sqrt{29}} \quad n = \frac{4}{\sqrt{29}}$

So the angle made with the x-axis is $\theta = \cos^{-1}\left(\frac{2}{\sqrt{29}}\right) = 68.2°$ (1 d.p.)

The angle made with the y-axis is $\theta = \cos^{-1}\left(\frac{3}{\sqrt{29}}\right) = 56.1°$ (1 d.p.)

The angle made with the z-axis is $\theta = \cos^{-1}\left(\frac{4}{\sqrt{29}}\right) = 42.0°$ (1 d.p.)

15 Two of the direction cosines are: $l = \cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and $n = \cos 60° = \frac{1}{2}$

To find the possible values of m use the fact that $l^2 + m^2 + n^2 = 1$

This gives $m^2 = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \Rightarrow m = \pm\frac{1}{2}$

So there are two solutions. In Cartesian equation form:

$$\frac{2x}{\sqrt{2}} = 2y = 2z \Rightarrow \frac{x}{\sqrt{2}} = y = z \text{ and } \frac{2x}{\sqrt{2}} = -2y = 2z \Rightarrow \frac{x}{\sqrt{2}} = -y = z$$

In vector equation form:

$$\mathbf{r} \times \begin{pmatrix}\sqrt{2}\\1\\1\end{pmatrix} = 0, \text{ which is equivalent to } \mathbf{r} \times \begin{pmatrix}1\\\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix} = 0$$

$$\text{and } \mathbf{r} \times \begin{pmatrix}\sqrt{2}\\-1\\1\end{pmatrix} = 0, \text{ which is equivalent to } \mathbf{r} \times \begin{pmatrix}1\\-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix} = 0$$

16 a The direction cosines satisfy $l = m = n$

As $l^2 + m^2 + n^2 = 1 \Rightarrow 3l^2 = 1 \Rightarrow l = m = n = \frac{1}{\sqrt{3}}$

Note that since all the direction cosines are equal the choice of sign does not matter.

16 b A Cartesian equation for the line is $\frac{x-1}{l}=\frac{y-2}{m}=\frac{z+1}{n}$

This gives $\frac{x-1}{\frac{1}{\sqrt{3}}}=\frac{y-2}{\frac{1}{\sqrt{3}}}=\frac{z+1}{\frac{1}{\sqrt{3}}}$

This simplifies to $x-1=y-2=z+1$

17 a The direction cosines are given by $l=\cos 75°, m=\cos 15°, n=0$

Using $\cos(A+B)=\cos A\cos B-\sin A\sin B$

$\cos 75°=\cos(45°+30°)=\cos 45°\cos 30°-\sin 45°\sin 30°$

$=\frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}=\frac{\sqrt{6}-\sqrt{2}}{4}$

Using $\cos(A-B)=\cos A\cos B+\sin A\sin B$

$\cos 15°=\cos(45°-30°)=\cos 45°\cos 30°+\sin 45°\sin 30°$

$=\frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}=\frac{\sqrt{6}+\sqrt{2}}{4}$

So in surd form, the direction cosines are $l=\frac{\sqrt{6}-\sqrt{2}}{4}, m=\frac{\sqrt{6}+\sqrt{2}}{4}, n=0$

Hence a vector equation for the line is $\left(\mathbf{r}-\begin{pmatrix}0\\0\\6\end{pmatrix}\right)\times\begin{pmatrix}\sqrt{6}-\sqrt{2}\\\sqrt{6}+\sqrt{2}\\0\end{pmatrix}=0$

b Expressing each line in a vector equation of the form $\mathbf{r}=\mathbf{a}+t\mathbf{b}$, for the wires to intersect requires finding λ and μ such that:

$$\begin{pmatrix}0\\0\\6\end{pmatrix}+\lambda\begin{pmatrix}\sqrt{6}-\sqrt{2}\\\sqrt{6}+\sqrt{2}\\0\end{pmatrix}=\begin{pmatrix}5\\2\\1\end{pmatrix}+\mu\begin{pmatrix}5-2(\sqrt{6}-\sqrt{2})\\2-2(\sqrt{6}+\sqrt{2})\\-5\end{pmatrix}$$

This gives:

$\left(\sqrt{6}-\sqrt{2}\right)\lambda=5+\left(5-2(\sqrt{6}-\sqrt{2})\right)\mu$ (1)

$\left(\sqrt{6}+\sqrt{2}\right)\lambda=2+\left(2-2(\sqrt{6}+\sqrt{2})\right)\mu$ (2)

$6=1-5\mu$ (3)

From (3): $\mu=-1$

From (1): $\left(\sqrt{6}-\sqrt{2}\right)\lambda=5-5+2\left(\sqrt{6}-\sqrt{2}\right)\Rightarrow\lambda=2$

This result satisfies (2): $2\left(\sqrt{6}+\sqrt{2}\right)=2-\left(2-2(\sqrt{6}+\sqrt{2})\right)=2\left(\sqrt{6}+\sqrt{2}\right)$

Hence the lines intersect.

c In reality the cable will not be completely horizontal but might have some slack due to gravity.

Challenge

a From the diagram, the point with spherical polar coordinates $\left(3, \frac{\pi}{4}, \frac{\pi}{3}\right)$ has a position vector:

$$r\sin\frac{\pi}{3}\cos\frac{\pi}{4}\mathbf{i} + r\sin\frac{\pi}{3}\sin\frac{\pi}{4}\mathbf{j} + r\cos\frac{\pi}{3}\mathbf{k}$$

So the direction cosines are:

$$l = \sin\frac{\pi}{3}\cos\frac{\pi}{4} = \frac{\sqrt{6}}{4}$$

$$m = \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{6}}{4}$$

$$n = \cos\frac{\pi}{3} = \frac{1}{2}$$

b In general a point on the line will have a position vector:

$$r\sin\varphi\cos\theta\mathbf{i} + r\sin\varphi\sin\theta\mathbf{j} + r\cos\varphi\mathbf{k}$$

So the direction cosines are:

$$l = \sin\varphi\cos\theta,\ m = \sin\varphi\sin\theta,\ n = \cos\varphi$$