Vectors 1C

1 For each problem, calculate the vector product in the bracket first and then perform the scalar product on the answer.

a $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 3 & 0 & 4 \end{vmatrix} = 4\mathbf{i}-\mathbf{j}-3\mathbf{k}$

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = (5\mathbf{i}+2\mathbf{j}-\mathbf{k}).(4\mathbf{i}-\mathbf{j}-3\mathbf{k})$$
$$= 20-2+3 = 21$$

b $\mathbf{c}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 4 \\ 5 & 2 & -1 \end{vmatrix} = -8\mathbf{i}+23\mathbf{j}+6\mathbf{k}$

$$\mathbf{b}.(\mathbf{c}\times\mathbf{a}) = (\mathbf{i}+\mathbf{j}+\mathbf{k}).(-8\mathbf{i}+23\mathbf{j}+6\mathbf{k})$$
$$= -8+23+6 = 21$$

c $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 3\mathbf{i}-6\mathbf{j}+3\mathbf{k}$

$$\mathbf{c}.(\mathbf{a}\times\mathbf{b}) = (3\mathbf{i}+4\mathbf{k}).(3\mathbf{i}-6\mathbf{j}+3\mathbf{k})$$
$$= 9+12 = 21$$

2 $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 2 & -3 & -5 \end{vmatrix} = -8\mathbf{i}+8\mathbf{j}-8\mathbf{k}$

$$\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = (\mathbf{i}-\mathbf{j}-2\mathbf{k}).(-8\mathbf{i}+8\mathbf{j}-8\mathbf{k})$$
$$= -8-8+16 = 0$$

If $\mathbf{a}.(\mathbf{b}\times\mathbf{c}) = 0$ then **a** is perpendicular to $\mathbf{b}\times\mathbf{c}$.

This means that **a** is parallel to the plane containing **b** and **c** (in fact $\mathbf{a} = \frac{1}{8}\mathbf{b}+\frac{3}{8}\mathbf{c}$).

3 $\overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix} = -2\mathbf{i}+\mathbf{j}+6\mathbf{k}$

Volume of the parallelepiped $= |\overrightarrow{AE}.(\overrightarrow{AB}\times\overrightarrow{AD})| = |(\mathbf{i}+\mathbf{j}+3\mathbf{k}).(-2\mathbf{i}+\mathbf{j}+6\mathbf{k})|$

$= |-2+1+18| = 17$

Alternatively, volume $= \begin{vmatrix} e_1 & e_2 & e_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix} = 1(0-2)-1(0-1)+3(6-0) = 17$

4 Let the vertices A, B, D and E have position vectors **a**, **b**, **d** and **e** respectively.

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = 4\mathbf{i} - 2\mathbf{k}$

$\overrightarrow{AD} = \mathbf{d} - \mathbf{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$\overrightarrow{AE} = \mathbf{e} - \mathbf{a} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$

Using the scalar triple product

$$\overrightarrow{AE}.\left(\overrightarrow{AB} \times \overrightarrow{AD}\right) = \begin{vmatrix} 3 & 1 & 1 \\ 4 & 0 & -2 \\ 3 & 2 & -1 \end{vmatrix} = 3(0+4) - 1(-4+6) + 1(8-0) = 12 - 2 + 8 = 18$$

So volume of parallelpiped = $\left|\overrightarrow{AE}.\left(\overrightarrow{AB} \times \overrightarrow{AD}\right)\right| = 18$

5 Let the vertices A, B, C and D have position vectors **a**, **b**, **c** and **d** respectively.

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \mathbf{j} - 2\mathbf{k}$

$\overrightarrow{AD} = \mathbf{d} - \mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\overrightarrow{AB}.\left(\overrightarrow{AC} \times \overrightarrow{AD}\right) = \begin{vmatrix} 3 & 1 & 1 \\ 0 & 1 & -2 \\ 2 & -1 & 1 \end{vmatrix} = 3(1-2) - 1(0+4) + 1(0-2) = -9$$

Volume of tetrahedron $= \frac{1}{6}\left|\overrightarrow{AB}.\left(\overrightarrow{AC} \times \overrightarrow{AD}\right)\right| = \frac{9}{6} = \frac{3}{2}$

6 a Let the vertices A, B, C and D have position vectors **a**, **b**, **c** and **d** respectively.

$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$\overrightarrow{BD} = \mathbf{d} - \mathbf{b} = 2\mathbf{j} + 2\mathbf{k}$

$$\overrightarrow{BC} \times \overrightarrow{BD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ 0 & 2 & 2 \end{vmatrix} = 2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$$

Area of face $BCD = \frac{1}{2}\left|\overrightarrow{BC} \times \overrightarrow{BD}\right| = \frac{1}{2}\sqrt{2^2 + 4^2 + (-4)^2} = \frac{\sqrt{36}}{2} = 3$

b The normal to the face BCD is in the direction of $\overrightarrow{BC} \times \overrightarrow{BD}$, i.e. in the direction $2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

As $|2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}| = \sqrt{2^2 + 4^2 + (-4)^2} = 6$

The unit vector normal to the face is $\frac{1}{6}(2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}) = \frac{1}{3}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$

Multiplying by –1 also gives a vector normal to the face BCD, so $-\frac{1}{3}(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ is a solution.

6 c $\overrightarrow{BA} = \mathbf{a} - \mathbf{b} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

$\overrightarrow{BA}.\left(\overrightarrow{BC} \times \overrightarrow{BD}\right) = (-\mathbf{i} + 3\mathbf{j} - \mathbf{k}).(2\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}) = -2 + 12 + 4 = 14$

Volume of tetrahedron $= \frac{1}{6}\left|\overrightarrow{AB}.\left(\overrightarrow{AC} \times \overrightarrow{AD}\right)\right| = \frac{14}{6} = \frac{7}{3}$

7 a Let the vertices A, B, C and D have position vectors **a**, **b**, **c** and **d** respectively. (Note **a** = 0.)
Now find the length of all edges, as a tetrahedron is regular if all of its edges are the same length.

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \mathbf{b}$, so $\left|\overrightarrow{AB}\right| = 2$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = \mathbf{c}$, so $\left|\overrightarrow{AC}\right| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\overrightarrow{AD} = \mathbf{d} - \mathbf{a} = \mathbf{d}$, so $\left|\overrightarrow{AD}\right| = \sqrt{1^2 + \left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3}\right)^2} = \sqrt{1 + \frac{3}{9} + \frac{24}{9}} = 2$

$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = -\mathbf{i} + \sqrt{3}\,\mathbf{j}$, so $\left|\overrightarrow{BC}\right| = \sqrt{(-1)^2 + (\sqrt{3}^2)} = 2$

$\overrightarrow{BD} = \mathbf{d} - \mathbf{b} = -\mathbf{i} + \frac{\sqrt{3}}{3}\mathbf{j} + \frac{2\sqrt{6}}{3}\mathbf{k}$, so $\left|\overrightarrow{BD}\right| = \sqrt{(-1)^2 + \left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3}\right)^2} = \sqrt{1 + \frac{3}{9} + \frac{24}{9}} = 2$

$\overrightarrow{CD} = \mathbf{d} - \mathbf{c} = \frac{-2\sqrt{3}}{3}\mathbf{j} + \frac{2\sqrt{6}}{3}\mathbf{k}$, so $\left|\overrightarrow{CD}\right| = \sqrt{\left(\frac{-2\sqrt{3}}{3}\right)^2 + \left(\frac{2\sqrt{6}}{3}\right)^2} = \sqrt{\frac{4}{3} + \frac{8}{3}} = 2$

All 6 edges have the same length and so the tetrahedron is regular.

b $\overrightarrow{AB}.\left(\overrightarrow{AC} \times \overrightarrow{AD}\right) = \begin{vmatrix} 2 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ 1 & \frac{\sqrt{3}}{3} & \frac{2\sqrt{6}}{3} \end{vmatrix} = 2\frac{2\sqrt{6}\sqrt{3}}{3} = \frac{4\sqrt{18}}{3} = 4\sqrt{2}$

Volume of tetrahedron $= \frac{1}{6}\left|\overrightarrow{AB}.\left(\overrightarrow{AC} \times \overrightarrow{AD}\right)\right| = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}$

8 a $\overrightarrow{AB} = (-\mathbf{i} + \mathbf{j} + 2\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

$\overrightarrow{AC} = (2\mathbf{i} - \mathbf{j} + \mathbf{k}) - (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 3 \\ 1 & -3 & 2 \end{vmatrix} = 7\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$

b Area of triangle ABC $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{7^2 + 7^2 + 7^2} = \frac{\sqrt{147}}{2} = \frac{7\sqrt{3}}{2}$

c $\overrightarrow{AO}.\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = (-\mathbf{i} - 2\mathbf{j} + \mathbf{k}).(7\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}) = -7 - 14 + 7 = -14$

Volume of tetrahedron $= \frac{1}{6}\left|\overrightarrow{AO}.\left(\overrightarrow{AB} \times \overrightarrow{AC}\right)\right| = \frac{14}{6} = \frac{7}{3}$

9 a $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{BC} = \mathbf{c} - \mathbf{b} = -3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ -3 & -1 & -2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 7\mathbf{k}$$

$\overrightarrow{BD} = \mathbf{d} - \mathbf{b} = -\mathbf{i} + 2\mathbf{k}, \quad \overrightarrow{DC} = \mathbf{c} - \mathbf{d} = -2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$

$$\overrightarrow{BD} \times \overrightarrow{DC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 2 \\ -2 & -1 & -4 \end{vmatrix} = 2\mathbf{i} - 8\mathbf{j} + \mathbf{k}$$

b i $\overrightarrow{AB} \times \overrightarrow{BC} = -\overrightarrow{BA} \times \overrightarrow{BC} \Rightarrow |\overrightarrow{AB} \times \overrightarrow{BC}| = |\overrightarrow{BA} \times \overrightarrow{BC}|$

So area of triangle $ABC = \frac{1}{2}|\overrightarrow{BA} \times \overrightarrow{BC}| = |\overrightarrow{AB} \times \overrightarrow{BC}|$

$$= \frac{1}{2}|-5\mathbf{i} + \mathbf{j} + 7\mathbf{k}| = \frac{1}{2}\sqrt{25 + 1 + 49}$$

$$= \frac{1}{2}\sqrt{75} = \frac{5\sqrt{3}}{2}$$

b ii $\overrightarrow{AB} \times \overrightarrow{BC} = -\overrightarrow{BA} \times \overrightarrow{BC} \Rightarrow \overrightarrow{BA} \times \overrightarrow{BC} = -\overrightarrow{AB} \times \overrightarrow{BC}$

$\overrightarrow{BD}.\left(\overrightarrow{BA} \times \overrightarrow{BC}\right) = (-\mathbf{i} + 2\mathbf{k}).(-5\mathbf{i} + \mathbf{j} + 7\mathbf{k}) = 5 + 14 = 19$

So volume of tetrahedron $ABCD = \frac{1}{6}\left|\overrightarrow{BD}.(\overrightarrow{BA} \times \overrightarrow{BC})\right| = \frac{19}{6}$

10 a $\mathbf{b} \times \mathbf{c} == \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 4 & -2 & 5 \end{vmatrix} = \mathbf{i} + 2\mathbf{j}$

As $\mathbf{a} = 2(\mathbf{b} \times \mathbf{c})$, $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{OQ}$ and to $\overrightarrow{OR}$, i.e. $\overrightarrow{OP}$ is perpendicular to the plane OQR.

b $|\overrightarrow{OP}| = |\mathbf{a}| = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$

Area of $OQR = \frac{1}{2}|\mathbf{b} \times \mathbf{c}| = \frac{1}{2}\sqrt{1^2 + 2^2} = \frac{\sqrt{5}}{2}$

Volume of tetrahedron $= \frac{1}{3} \times \text{base} \times \text{height} = \frac{1}{3} \times \frac{\sqrt{5}}{2} \times 2\sqrt{5} = \frac{5}{3}$

c Using the scalar triple product:

$$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 4 & 0 \\ 2 & -1 & 3 \\ 4 & -2 & 5 \end{vmatrix} = 2(-5 + 6) - 4(10 - 12) = 2 + 8 = 10$$

$\mathbf{a}.(\mathbf{b} \times \mathbf{c})$ is $6 \times$ volume of tetrahedron (from part **b**), so result verified.

11 a $\overrightarrow{OB}\times\overrightarrow{OC}=(2\mathbf{i}-\mathbf{j}-4\mathbf{k})\times(-2\mathbf{i}+4\mathbf{j}-2\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & -4\\ -2 & 4 & -2\end{vmatrix}=18\mathbf{i}+12\mathbf{j}+6\mathbf{k}$

b $\overrightarrow{OA}.\left(\overrightarrow{BA}\times\overrightarrow{BC}\right)=(3\mathbf{i}+2\mathbf{j}+\mathbf{k}).(18\mathbf{i}+12\mathbf{j}+6\mathbf{k})=54+24+6=84$

So volume in design units $=\frac{1}{6}\left|\overrightarrow{OA}.(\overrightarrow{OB}\times\overrightarrow{OC})\right|=\frac{84}{6}=14$

Prototype volume = volume in design units $\times 2^3=14\times 8=112\,\text{cm}^3$

$\Rightarrow$ Mass of a prototype model $=112\times 1.13=127$ g (to the nearest gram)

12 By drawing a rough sketch, the three non-parallel edges of the parallelepiped that join at the origin are given by the vectors 0.6**i** + 0.6**j**, 0.9**i** – 0.9**j**, and –0.4**i** – 0.4**j** – 1.3**k**

The volume in the scientist's scale $=\begin{vmatrix}0.6 & 0.6 & 0\\ 0.9 & -0.9 & 0\\ -0.4 & -0.4 & -1.3\end{vmatrix}=0.6(-0.9\times -1.3)-0.6(0.9\times -1.3)=1.404$

1 angstrom (Å) = 0.1 nanometres, so 1 nanometre = 10 Å, and 1 cubic nanometre = 10^3 Å

So volume unit cell of crystal in cubic angstroms $=1.404\times 10^3=1400\,\text{Å}$ (2 s.f.)

13 a The volume of the parallelepiped $=\left|\overrightarrow{AB}.\left(\overrightarrow{BD}\times\overrightarrow{BC}\right)\right|$

The volume of the tetrahedron $=\frac{1}{6}\left|\overrightarrow{EC}.\left(\overrightarrow{EM}\times\overrightarrow{NC}\right)\right|$

Now from the diagram:

$\overrightarrow{EC}=\overrightarrow{AB}$

$\overrightarrow{EM}=\frac{1}{2}\overrightarrow{BD}$

$\overrightarrow{NC}=\overrightarrow{NB}+\overrightarrow{BC}$

Using these results and the fact that both the vector product and the scalar product are distributive over vector addition, this gives:

Volume of the tetrahedron $=\frac{1}{12}\left|\overrightarrow{AB}.\left(\overrightarrow{BD}\times\left(\overrightarrow{NB}+\overrightarrow{BC}\right)\right)\right|$

$=\frac{1}{12}\left|\overrightarrow{AB}.\left(\left(\overrightarrow{BD}\times\overrightarrow{NB}\right)+\left(\overrightarrow{BD}\times\overrightarrow{BC}\right)\right)\right|$ as vector product distributive

$=\frac{1}{12}\left|\overrightarrow{AB}.\left(\overrightarrow{BD}\times\overrightarrow{NB}\right)+\overrightarrow{AB}.\left(\overrightarrow{BD}\times\overrightarrow{BC}\right)\right|$ as scalar product distributive

But $\overrightarrow{BD}\times\overrightarrow{NB}$ is perpendicular to $\overrightarrow{AB}$ so $\overrightarrow{AB}.\left(\overrightarrow{BD}\times\overrightarrow{NB}\right)=0$

So the expression for the volume of the tetrahedron simplifies to $\frac{1}{12}\left|\overrightarrow{AB}.\left(\overrightarrow{BD}\times\overrightarrow{BC}\right)\right|$

Hence the ratio of the two volumes is $12:1$

b The ratio remains unchanged since the argument in part **a** does not use any data about N other than it lies on the line AB.

14 Split the pyramid into two tetrahedrons *AEDB* and *CEDB* so the volume of the pyramid is just the combined volume of the two tetrahedrons.

$\overrightarrow{AD} = \mathbf{i} + 2\mathbf{k}, \quad \overrightarrow{AB} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{AE} = 4\mathbf{i} + \mathbf{k}$

$$\overrightarrow{AD}.\left(\overrightarrow{AB} \times \overrightarrow{AE}\right) = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 4 & 0 & 1 \end{vmatrix} = 2 - 16 = -14$$

Volume of *AEDB* $= = \frac{1}{6}|-14| = \frac{14}{6} = \frac{7}{3}$

$\overrightarrow{CB} = -\mathbf{i} - 2\mathbf{k}, \quad \overrightarrow{CD} = -\mathbf{i} - 2\mathbf{j} - \mathbf{k}, \quad \overrightarrow{CE} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$

$$\overrightarrow{CB}.\left(\overrightarrow{CD} \times \overrightarrow{CE}\right) = \begin{vmatrix} -1 & 0 & -2 \\ -1 & -2 & -1 \\ 2 & -2 & -2 \end{vmatrix} = -2 - 12 = -14$$

Volume of *CEDB* $= = \frac{1}{6}|-14| = \frac{14}{6} = \frac{7}{3}$

Hence the combined volume $= \frac{7}{3} + \frac{7}{3} = \frac{14}{3}$

Challenge

a Let $a = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad b = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \quad c = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$

$$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

$$= a_1b_2\mathbf{k} - a_1b_3\mathbf{j} - a_2b_1\mathbf{k} + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} - a_3b_2\mathbf{i} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$(\mathbf{a} \times \mathbf{b}).\mathbf{c} = \left((a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\right).(c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k})$$

$$= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3$$

$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

So $\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}).\mathbf{c}$

b $\mathbf{d}.(\mathbf{a} \times (\mathbf{b} + \mathbf{c})) = (\mathbf{d} \times \mathbf{a}).(\mathbf{b} + \mathbf{c})$ applying part **a**

$= (\mathbf{d} \times \mathbf{a}).\mathbf{b} + (\mathbf{d} \times \mathbf{a}).\mathbf{c}$ scalar product is distributive over vector addition

$= \mathbf{d}.(\mathbf{a} \times \mathbf{b}) + \mathbf{d}.(\mathbf{a} \times \mathbf{c})$ applying part **a**

$= \mathbf{d}.(\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c})$ scalar product is distributive over vector addition

c Since the equality holds for any choice of vector **d**, it follows that $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$