Exam-style practice: Paper 1

1 $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{2\sec^2 t}{2\sin t \cos t} = \dfrac{1}{\sin t \cos^3 t} = \operatorname{cosec} t \sec^3 t$

2 a $2(7x-5)-6x<10x-7$

$\Rightarrow 14x-10-6x<10x-7$

$\Rightarrow -3<2x$

So $x>-\dfrac{3}{2}$

b $|2x+5|-3>0$

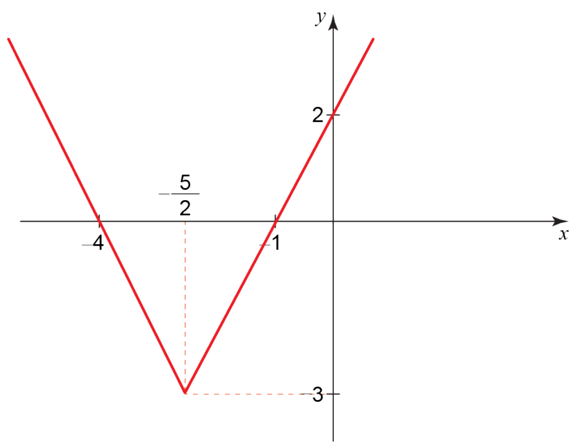
Solve $|2x+5|-3=0$

$x<-\dfrac{5}{2}$: $-(2x+5)-3=0$

$\Rightarrow 2x=-8 \Rightarrow x=-4$

$x>-\dfrac{5}{2}$: $(2x+5)-3=0$

$\Rightarrow 2x=-2 \Rightarrow x=-1$



From the graph, we see that the inequality holds when $x<-4$ or $x>-1$

c For both inequalities to hold, x must satisfy both $x>-\frac{3}{2}$ and $x<-4$ or $x>-1$ so the solution is $x>-1$

3 a $2x+y-3=0 \Rightarrow y=3-2x$

Substitute this equation for y in the equation of the circle

$x^2+kx+(3-2x)^2+4(3-2x)=4$

$5x^2+kx-20x+17=0$

If this equation has solutions, the line will intersect the circle. As the equation is a positive quadratic, there will be no solutions, and the line will not intersect the circle, if

$5x^2+kx-20x+17>0$

b As there are no solutions to the equation

$5x^2+(k-20)x+17=0$

the discriminant must be less than zero

$\Rightarrow (k-20)^2-4(5)(17)<0$

$\Rightarrow k^2-40k+400-340<0$

$\Rightarrow k^2-40k+60<0$

Solve $k^2-40k+60=0$

Using the quadratic formula,

$$k=\frac{40\pm\sqrt{(-40)^2-4(1)(60)}}{2(1)}=20\pm2\sqrt{85}$$

Since $f(k)=k^2-40k+60$ is a positive quadratic in k, the set of values of k for which $f(k)$ is negative must be

$20-2\sqrt{85}<k<20+2\sqrt{85}$

4 Let $f(\theta)=\cos\theta$

$$\begin{aligned}
f'(\theta) &= \lim_{h\to0}\frac{f(\theta+h)-f(\theta)}{h} \\
&= \lim_{h\to0}\frac{\cos(\theta+h)-\cos\theta}{h} \\
&= \lim_{h\to0}\frac{\cos\theta\cos h-\sin\theta\sin h-\cos\theta}{h} \\
&= \lim_{h\to0}\left[\left(\frac{\cos h-1}{h}\right)\cos\theta-\left(\frac{\sin h}{h}\right)\sin\theta\right] \\
&= -\sin\theta
\end{aligned}$$

5 a Using the binomial expansion, the coefficient of x^2 in the expansion of $(3+px)^6$ is

$$\frac{6(6-1)}{2!}(3)^4 p^2 = 1215p^2$$

As $1215p^2 = 19440 \Rightarrow p^2 = 16$

So solutions are $p = 4$, $p = -4$

b The coefficient of x^5 is

$$\frac{6(6-1)(6-2)(6-3)(6-4)}{5!}3p^5 = 1215p^2$$

Since this negative, use $p = -4$, so the coefficient is

$$6 \times 3 \times (-4)^5 = -18432$$

6 First find the y-coordinate of R

$y = (2)^2 + 4(2) - 2 = 10$ so $R(2,10)$

To find the normal line to the curve at R, find the gradient at R

$$\frac{dy}{dx} = 2x + 4, \text{ so at } x = 2, \frac{dy}{dx} = 8$$

At R, the normal line will therefore have a gradient of $-\frac{1}{8}$

So the equation of the normal line at R is

$$(y-10) = -\frac{1}{8}(x-2) \Rightarrow y = -\frac{x}{8} + \frac{41}{4}$$

To find T, solve

$$-\frac{x}{8} + \frac{41}{4} = x^2 + 4x - 2$$

$$\Rightarrow 8x^2 + 33x - 98 = 0$$

In factorising this equation, remember that $x = 2$ is a solution

So $(x-2)(8x+49) = 0$

The normal also meets the curve at $x = -\frac{49}{8}$

And when $y = -\frac{-\frac{49}{8}}{8} + \frac{41}{4} = \frac{705}{64}$

Required coordinates are $\left(-\frac{49}{8}, \frac{705}{64}\right)$

7 a $u_1 = a, u_2 = ar = 96, S_\infty = 600$

$$a = \frac{96}{r}, S_\infty = \frac{\frac{96}{r}}{1-r} = 600$$

$$\Rightarrow 600 - 600r = \frac{96}{r}$$

$$\Rightarrow 600r^2 - 600r + 96 = 0$$

$$\Rightarrow 25r^2 - 25r + 4 = 0 \quad \text{(dividing by 24)}$$

b $25r^2 - 25r + 4 = 0$

Factorise $(5r - 1)(5r - 4) = 0$

Or use the quadratic formula

$$r = \frac{-(-25) \pm \sqrt{(-25)^2 - 4(4)(25)}}{2(25)}$$

$$= \frac{25 \pm 15}{50}$$

Solutions are $r = \frac{1}{5} = 0.2, r = \frac{4}{5} = 0.8$

c The larger value of r is $r = 0.8$. The corresponding value of a is

$$a = \frac{96}{0.8} = 120$$

d $S_n = \frac{a(1-r^n)}{1-r} > 599.9$

$$\Rightarrow \frac{120\left(1-(0.8)^n\right)}{1-0.8} > 599.9$$

$$\Rightarrow 1-(0.8)^n > \frac{599.9}{600}$$

$$\Rightarrow (0.8)^n < \frac{0.1}{600}$$

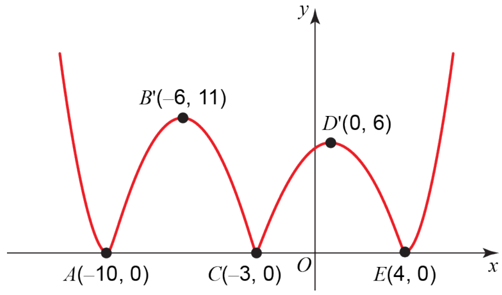
$$\Rightarrow n \ln 0.8 < \ln\frac{0.1}{600}$$

$$\Rightarrow n > \frac{\ln\frac{0.1}{600}}{\ln 0.8} \quad \text{(as ln0.8 is negative)}$$

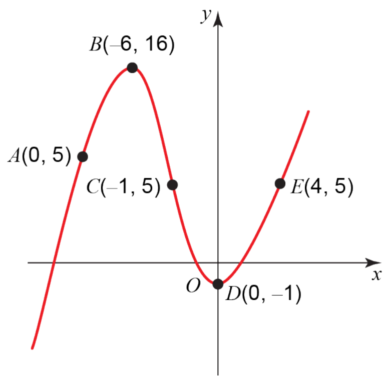
$$\Rightarrow n > 38.986 \text{ (3 d.p.)}$$

So $n = 39$

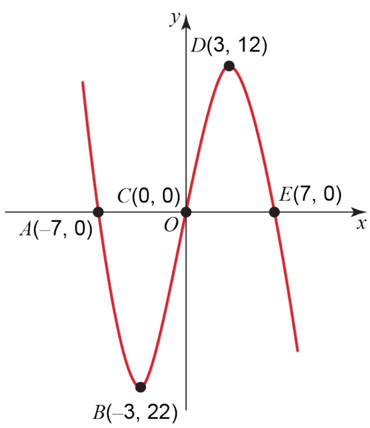
8 a Reflect graph of f(x) the x-axis in regions where f(x) < 0, i.e. $-10 < x < -3$ and $x > 4$



b First reflect the graph in the x-axis to obtain $y = -\mathrm{f}(x)$ then translate this graph by the vector $\begin{pmatrix}0\\5\end{pmatrix}$



c Translate by the vector $\begin{pmatrix}3\\0\end{pmatrix}$ to obtain $y = \mathrm{f}(x - 3)$ and then stretch in the y-direction by a scale factor of 2



9 $31 - 25\cos x = 19 - 12\sin^2 x$

$\Rightarrow 31 - 25\cos x = 19 - 12(1 - \cos^2 x)$

$\Rightarrow 12\cos^2 x + 25\cos x - 24 = 0$

$$\Rightarrow \cos x = \frac{-25 \pm \sqrt{(25)^2 - 4(12)(-24)}}{2(12)}$$

$= 0.7148, -2.7981$ (4 d.p.)

$\cos x = -2.798...$ has no solutions,

since $|\cos x| \leq 1$

So there are two solutions in the required interval

$\cos^{-1}(0.7148) = 0.77$ (2 d.p.)

and $2\pi - \cos^{-1}(0.7148) = 5.51$ (2 d.p.)

10 a Let the constant of proportionality be $-k$, where $k > 0$. Therefore

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -kV \Rightarrow V = C\mathrm{e}^{-kt}$$

where C is a constant

$V = V_0$ at $t = 0 \Rightarrow C = V_0$

So $V = V_0\mathrm{e}^{-kt}$

b $25000 = V_0\mathrm{e}^{-2k}$ (1)

$15000 = V_0\mathrm{e}^{-5k}$ (2)

Dividing equations (1) and (2)

$\frac{5}{3} = \mathrm{e}^{3k} \Rightarrow k = \frac{1}{3}\ln\left(\frac{5}{3}\right)$

Substituting this value of k into (1)

$V_0 = 25000\mathrm{e}^{\frac{2}{3}\ln\frac{5}{3}} = 35143.0$ (6 s.f.)

So $V_0 = 35\,100$ to the nearest hundred

c $V_0\mathrm{e}^{-kt} = 5000$

$$\Rightarrow \mathrm{e}^{kt} = \frac{V_0}{5000}$$

$$\Rightarrow kt = \ln\left(\frac{V_0}{5000}\right)$$

$$\Rightarrow t = 3\frac{\ln\left(\frac{35143}{5000}\right)}{\ln\left(\frac{5}{3}\right)} = 11.45 \text{ years (2 d.p.)}$$

d k should be changed to a smaller value e.g. 0.1 (any value smaller than 0.17 acceptable)

11 a Apply the cosine rule to the triangles ABC and ACD to find $\angle BCA$ and $\angle ACD$

$$\cos(\angle BCA) = \frac{21^2 + 19^2 - 8^2}{2(21)(19)}$$
$$= 0.9248 \text{ (4 d.p.)}$$
$$\Rightarrow \angle BCA = 0.3903 \text{ rad (4 d.p.)}$$
$$\cos(\angle ACD) = \frac{14^2 + 21^2 - 11^2}{2(14)(21)}$$
$$= 0.8776 \text{ (4 d.p.)}$$
$$\Rightarrow \angle ACD = 0.5001 \text{ rad (4 d.p.)}$$

So $\angle BCD = \angle BCA + \angle ACD = 0.890$ rad

Now apply the cosine rule to triangle BCD

$$\cos(\angle BCD) = \frac{14^2 + 19^2 - |BD|^2}{2(14)(19)}$$
$$|BD| = \sqrt{14^2 + 19^2 - 2(14)(19)\cos(\angle BCD)}$$
$$= \sqrt{196 + 361 - 334.847} = 14.9 \text{ (1 d.p.)}$$

b The shortest distance between two points is a straight line, so any other route will be longer.

12 a $y = -0.01x^2 + 0.22x + 1.58$
$$= -0.01(x^2 - 22x - 158)$$
$$= -0.01((x-11)^2 - 279)$$
$$= 2.79 - 0.01(x-11)^2$$

b The ball reaches its highest point when its horizontal distance from the goal is 11 metres. Its maximum height is 2.79 metres.

c The ball is kicked when $y = 0$

$$2.79 - 0.01(x-11)^2 = 0$$
$$\Rightarrow (x-11)^2 = \frac{2.79}{0.01} = 279$$
$$\Rightarrow x - 11 = \pm 16.703$$

$x > 0$, so $x = 27.7$ m (1 d.p.)

d At $x = 0$, $y = 2.79 - 0.01(-11)^2 = 1.58$

As $1.5 < 1.58 < 2.44$, the ball will go not be saved by the keeper but it will go under the crossbar, so it will enter the goal

13 a Surface area of box
$$= 2x^2 + 2(2xh + xh) = 2x^2 + 6xh$$
Surface area of lid
$$= 2x^2 + 2(6x + 3x) = 2x^2 + 18x$$
Total surface area
$$= 4x^2 + 6xh + 18x = 5356$$
So $h = \dfrac{5356 - 18x - 4x^2}{6x}$

$$V = 2x^2 h = \frac{2}{3}\left(2678x - 9x^2 - 2x^3\right)$$

b $V = \dfrac{2}{3}(2678x - 9x^2 - 2x^3)$

$$\Rightarrow \frac{dV}{dx} = \frac{2}{3}(2678 - 18x - 6x^2)$$

$\dfrac{dV}{dx} = 0$ at a stationary point so

$$6x^2 + 18x - 2678 = 0$$

Since $x > 0$

$$x = \frac{-18 + \sqrt{(18)^2 - 4(6)(-2678)}}{2(6)}$$
$$= 19.68 \text{ cm (2 d.p.)}$$

c $\dfrac{d^2V}{dx^2} = \dfrac{2}{3}(-18 - 12x)$

Since $x > 0$, $\dfrac{d^2V}{dx^2} < 0 \Rightarrow$ maximum

d $x = 19.68 \Rightarrow V = 22\,648.7 \text{ cm}^3$ (1 d.p.)

e From part **a**,
surface area of lid $= 2x^2 + 18x$
So percentage of cardboard in the lid is

$$\frac{2(19.68)^2 + 18(19.68)}{5356} \times 100$$
$$= 21.1\% \text{ (1 d.p.)}$$