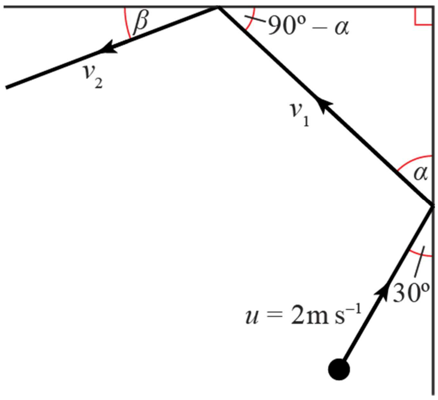
Elastic collisions in two dimensions 5B

1



a First collision: $e = 0.5$

For motion parallel to the wall:

$v_1 \cos\alpha = 2\cos 30°$ **(1)**

For motion perpendicular to the wall:

$v_1 \sin\alpha = 0.5 \times 2\sin 30°$ **(2)**

Squaring and adding equations **(1)** and **(2)** gives:

$$v_1^2 \cos^2\alpha + v_1^2 \sin^2\alpha = 4\cos^2 30° + \sin^2 30°$$

$$v_1^2(\cos^2\alpha + \sin^2\alpha) = 4 \times \frac{3}{4} + \frac{1}{4} = \frac{13}{4}$$

$$v_1 = \frac{\sqrt{13}}{2} = 1.80\,\text{m s}^{-1} \text{ (3 s.f.)}$$

Dividing equation **(2)** by equation **(1)** gives:

$$\tan\alpha = 0.5\tan 30° = \frac{1}{2\sqrt{3}}$$

$$\Rightarrow \alpha = 16.1° \text{ (3 s.f.)}$$

1 b Second collision: $e = 0.5$

For motion parallel to the wall:

$v_2 \cos\beta = v_1 \cos(90° - \alpha) = v_1 \sin\alpha$ **(3)**

For motion perpendicular to the wall:

$v_2 \sin\beta = 0.5 \times v_1 \sin(90° - \alpha) = 0.5 v_1 \cos\alpha$ **(4)**

Squaring and adding equations **(3)** and **(4)** gives:

$v_2^2 \cos^2\beta + v_2^2 \sin^2\beta = v_1^2 \sin^2\alpha + 0.25 v_1^2 \cos^2\alpha$

As $\tan\alpha = \dfrac{1}{2\sqrt{3}}$, by Pythagoras $\sin\alpha = \dfrac{1}{\sqrt{13}}$ and $\cos\alpha = \dfrac{2\sqrt{3}}{\sqrt{13}}$

$$v_2^2(\cos^2\beta + \sin^2\beta) = v_1^2\left(\frac{1}{13} + \frac{3}{13}\right) = \frac{13}{4} \times \frac{4}{13} = 1$$

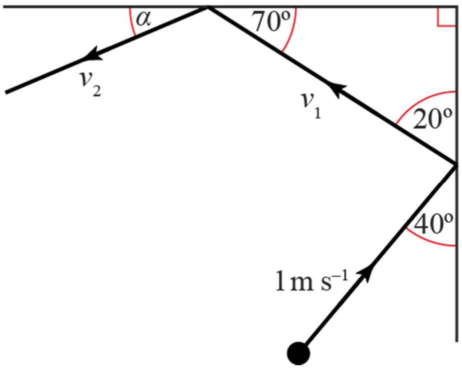
$v_2 = 1\,\text{m s}^{-1}$

Dividing equation **(4)** by equation **(3)** gives:

$$\tan\beta = \frac{0.5}{\tan\alpha} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

$\Rightarrow \beta = 60°$

2



a First collision – for motion parallel to the wall:

$v_1 \cos 20° = \cos 40°$ **(1)**

$$v_1 = \frac{\cos 40°}{\cos 20°} = 0.815\,\text{m}\,\text{s}^{-1} \text{ (3 s.f.)}$$

b For motion perpendicular to the wall:

$v_1 \sin 20° = e \sin 40°$ **(2)**

Dividing equation **(2)** by equation **(1)** gives:

$\tan 20° = e \tan 40°$

$$\Rightarrow e = \frac{\tan 20°}{\tan 40°} = 0.434 \text{ (3 s.f.)}$$

c Second collision – for motion parallel to the wall:

$v_2 \cos \alpha = v_1 \cos 70°$ **(3)**

For motion perpendicular to the wall:

$v_2 \sin \alpha = e v_1 \sin 70°$ **(4)**

Dividing equation **(4)** by equation **(3)** gives:

$$\tan \alpha = e \tan 70° = \frac{\tan 20° \tan 70°}{\tan 40°}$$

but $\tan 20° \tan 70° = 1$

$$\text{So } \tan \alpha = \frac{1}{\tan 40°} = \tan(90° - 40°) = \tan 50°$$

$\Rightarrow \alpha = 50°$

Substituting into equation **(3)** gives:

$v_2 \cos 50° = v_1 \cos 70°$

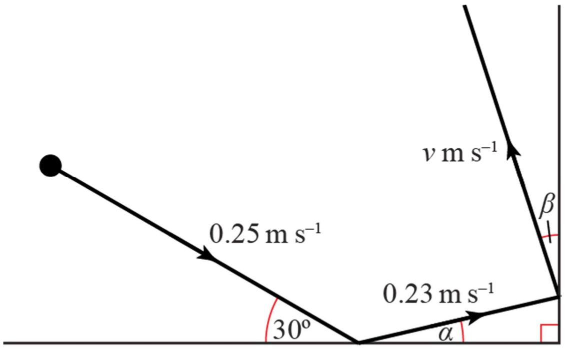
$$v_2 \cos 50° = \frac{\cos 40°}{\cos 20°} \cos 70°$$

but $\cos 50° = \sin 40°$ and $\cos 70° = \sin 20°$

$$\text{So } v_2 = \frac{\cos 40° \sin 20°}{\cos 20° \sin 40°}$$

$$v_2 = \frac{\tan 20°}{\tan 40°} = 0.434\,\text{m}\,\text{s}^{-1} \text{ (3 s.f.)}$$

3



a First collision – let the angle the sphere makes with the wall after the collision be α

For motion parallel to the wall:

$0.23\cos\alpha = 0.25\cos 30°$ **(1)**

$$\cos\alpha = \frac{0.25}{0.23}\times\frac{\sqrt{3}}{2} = 0.94133\ldots$$

$\Rightarrow \alpha = 19.7°$ (3 s.f.)

After the impact with the first wall, the sphere moves at 19.7° to the wall

b For motion perpendicular to the wall:

$0.23\sin\alpha = 0.25e\sin 30°$ **(2)**

Dividing equation **(2)** by equation **(1)** gives:

$\tan\alpha = e\tan 30°$

$$\Rightarrow e = \frac{\tan\alpha}{\tan 30°} = \frac{\tan 19.72°}{\tan 30°} = 0.621 \text{ (3 s.f.)}$$

c First collision – let the angle the sphere makes with the wall after the collision be β and its speed after the collision be v

For motion parallel to the wall:

$v\cos\beta = 0.23\cos(90° - \alpha) = 0.23\sin\alpha$ **(3)**

For motion perpendicular to the wall:

$v\sin\beta = 0.23e\sin(90° - \alpha) = 0.23e\cos\alpha$ **(4)**

Squaring and adding equations **(3)** and **(4)** gives:

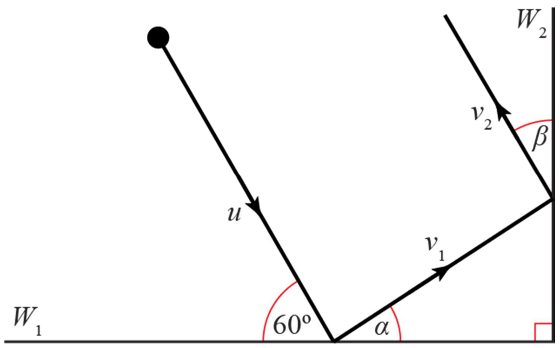
$$v^2\cos^2\beta + v^2\sin^2\beta = 0.23^2\sin^2\alpha + 0.23^2e^2\cos^2\alpha$$

$$v^2(\cos^2\beta + \sin^2\beta) = 0.23^2(1 - \cos^2\alpha + e^2\cos^2\alpha)$$

$$v^2 = 0.0529(1 - 0.94133^2 + 0.621^2\times 0.94133^2) = 0.0241\ldots$$

Kinetic energy after second collision $= \frac{1}{2}mv^2 = \frac{1}{2}\times 0.1\times 0.0241 = 0.00121$ J (3 s.f.)

4



a First collision – let the angle the sphere makes with the wall after the collision be α, its speed before impact be u and its speed after impact v_1

Kinetic energy before first collision $= \frac{1}{2} \times 2 \times u^2$

So $u^2 = 9 \Rightarrow u = 3\,\text{m}\,\text{s}^{-1}$

For motion parallel to the wall:

$v_1 \cos\alpha = 3\cos 60° = 1.5$ **(1)**

For motion perpendicular to the wall:

$v_1 \sin\alpha = 0.75 \times 3 \sin 60° = 1.125\sqrt{3}$ **(2)**

Squaring and adding equations **(1)** and **(2)** gives:

$v_1^2 \cos^2\alpha + v_1^2 \sin^2\alpha = 2.25 + 3.7969$

$v_1^2(\cos^2\alpha + \sin^2\alpha) = 6.0469$

$v_1 = 2.46\,\text{m}\,\text{s}^{-1}$ (3 s.f.)

Dividing equation **(2)** by equation **(1)** gives:

$\tan\alpha = 0.75\tan 60° = 0.75\sqrt{3}$

$\Rightarrow \alpha = 52.4°$ (3 s.f.)

b Second collision – for motion parallel to the wall:

$v_2 \cos\beta = v_1 \cos(90° - \alpha) = v_1 \sin\alpha$ **(3)**

For motion perpendicular to the wall:

$v_2 \sin\beta = 0.6 v_1 \sin(90° - \alpha) = 0.6 v_1 \cos\alpha$ **(4)**

Squaring and adding equations **(3)** and **(4)** gives:

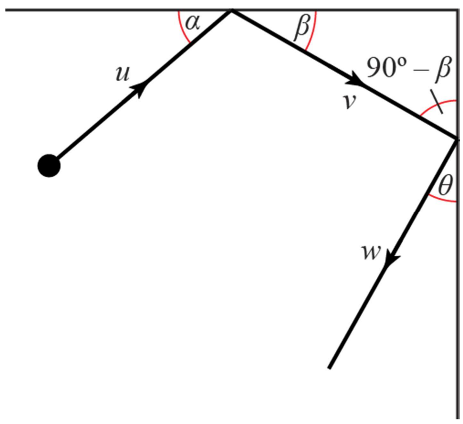
$v_2^2 \cos^2\beta + v_2^2 \sin^2\beta = v_1^2 \sin^2\alpha + v_1^2 0.36\cos^2\alpha$

$v_2^2(\cos^2\beta + \sin^2\beta) = v_1^2(\sin^2\alpha + 0.36\cos^2\alpha)$

$v_2^2 = 6.0469(0.6279 + 0.13395) = 4.60685$

Kinetic energy after second collision $= \frac{1}{2} m v_2^2 = \frac{1}{2} \times 2 \times 4.60685 = 4.61$ J (3 s.f.)

5



First collision

For motion parallel to the wall:

$v\cos\beta = u\cos\alpha$ **(1)**

For motion perpendicular to the wall:

$v\sin\beta = eu\sin\alpha$ **(2)**

Dividing equation **(2)** by equation **(1)** gives:

$\tan\beta = e\tan\alpha$

Second collision

For motion parallel to the wall:

$w\cos\theta = v\cos(90° - \beta) = v\sin\beta$ **(3)**

For motion perpendicular to the wall:

$w\sin\theta = ev\sin(90° - \beta) = ev\cos\beta$ **(4)**

Dividing equation **(4)** by equation **(3)** gives:

$$\tan\theta = \frac{e}{\tan\beta} = \frac{e}{e\tan\alpha}$$

$$= \frac{1}{\tan\alpha} = \cot\alpha = \tan(90° - \alpha)$$

So $\theta = 90° - \alpha$, which means that after the collision with the second wall the sphere's path is parallel to its original path but in the opposite direction.

Substituting for θ in equation **(3)** gives:

$w\cos\theta = w\cos(90° - \alpha) = v\sin\beta$

$\Rightarrow w\sin\alpha = v\sin\beta$

So from equation **(2)**

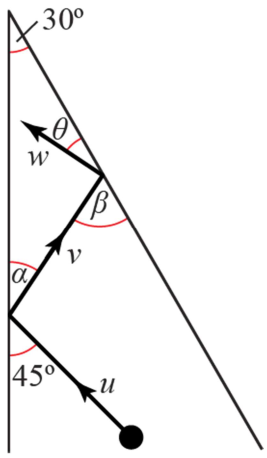
$w\sin\alpha = v\sin\beta = eu\sin\alpha$

$\Rightarrow w = eu$, the speed after the second collision

6 First impact:

For motion parallel to the wall:

$$v\cos\alpha = u\cos 45° = \frac{u}{\sqrt{2}} \qquad \textbf{(1)}$$



For motion perpendicular to the wall:

$$v\sin\alpha = eu\sin 45° = \frac{u}{\sqrt{3}\sqrt{2}} \qquad \textbf{(2)}$$

Squaring and adding equations **(1)** and **(2)** gives:

$$v^2\cos^2\alpha + v^2\sin^2\alpha = u^2\left(\frac{1}{2}+\frac{1}{6}\right)$$

$$v^2 = \frac{2u^2}{3}$$

Dividing equation **(2)** by equation **(1)** gives:

$$\tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = 30°$$

So $180° - 30° - 30° = 180° - \beta \Rightarrow \beta = 60°$

Second impact:

For motion parallel to the wall:

$$w\cos\theta = v\cos 60° = \frac{v}{2} \qquad \textbf{(3)}$$

For motion perpendicular to the wall:

$$w\sin\theta = ev\sin 60° = \frac{v}{2} \qquad \textbf{(4)}$$

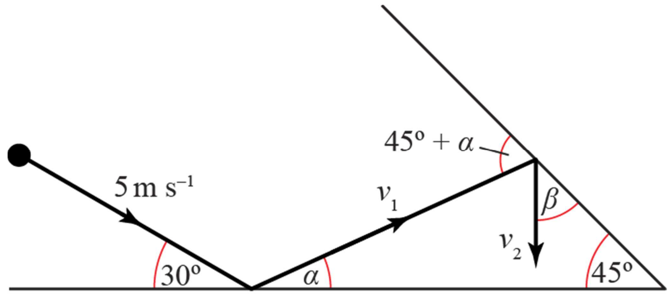
Squaring and adding equations **(3)** and **(4)** gives:

$$w^2\cos^2\theta + w^2\sin^2\theta = v^2\left(\frac{1}{4}+\frac{1}{4}\right)$$

$$w^2 = \frac{v^2}{2} = \frac{u^2}{3}$$

$$\Rightarrow w = \frac{u}{\sqrt{3}} = \frac{\sqrt{3}u}{3}\,\mathrm{m\,s^{-1}}$$

7



a First impact:
For motion parallel to the wall:

$v_1 \cos\alpha = 5\cos 30°$ **(1)**

For motion perpendicular to the wall:

$v_1 \sin\alpha = 0.8 \times 5 \sin 30°$ **(2)**

Squaring and adding equations **(1)** and **(2)** gives:

$$v_1^2 \cos^2\alpha + v_1^2 \sin^2\alpha = 25 \times \frac{3}{4} + 16 \times \frac{1}{4}$$

$$v_1^2 = \frac{91}{4} \Rightarrow v = 4.77\,\text{m}\,\text{s}^{-1}\ \text{(3 s.f.)}$$

Dividing equation **(2)** by equation **(1)** gives:

$$\tan\alpha = 0.8\tan 30° = \frac{4}{5\sqrt{3}} \Rightarrow \alpha = 24.8°\ \text{(3 s.f.)}$$

b Second impact:
For motion parallel to the wall:

$v_2 \cos\beta = v_1 \cos(45° + \alpha)$ **(3)**

For motion perpendicular to the wall:

$v_2 \sin\beta = 0.8 v_1 \sin(45° + \alpha)$ **(4)**

Dividing equation **(4)** by equation **(3)** gives:

$$\tan\beta = 0.8\tan(45° + \alpha) = 0.8\tan(45° + 24.791°) = 2.1733$$

$$\Rightarrow \beta = 65.3°\ \text{(3 s.f.)}$$

From equation **(3)**

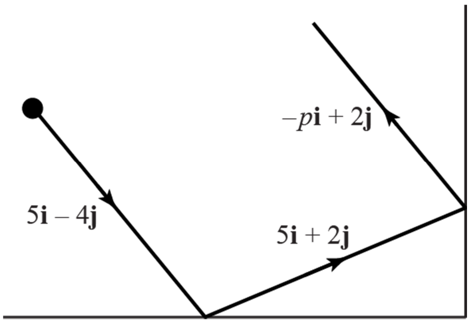
$$v_2 \cos 65.291° = v_1 \cos(45° + \alpha)$$

$$v_2 = 4.77 \times \frac{\cos 69.791°}{\cos 65.291°} = 3.94\,\text{m}\,\text{s}^{-1}\ \text{(3 s.f.)}$$

c If $e_2 > 0.8$ and e_1 remains unchanged, $(45° + \alpha)$ and v_1 do not change; however, $\tan\beta$ increases, so β increases and v_2 (which depends on v_1 and $\cos\beta$) increases.

So the velocity of the sphere after the second collision would be greater and the angle it makes with the wall after the collision would be greater.

8



a Since first collision causes no change to the **i** component, first collision must be with the wall parallel to this vector.

Considering **j** components only for the first collision:
$2 = 4e \Rightarrow e = 2$

Considering only the **i** component changes for the second collision:
$5 = ep = 2p \Rightarrow p = 2.5$

$$\begin{aligned}\text{Loss of kinetic energy} &= \frac{1}{2}m\left(|\mathbf{u}|^2 - |\mathbf{v}|^2\right)\\ &= \frac{1}{2}m\left((5^2+4^2)-(2.5^2+2^2)\right)\\ &= \frac{1}{2}m\left(41-\frac{41}{4}\right)\\ &= \frac{123}{8}m \text{ J}\end{aligned}$$

b Initial velocity, $\mathbf{u} = 5\mathbf{i} - 4\mathbf{j}$ and final velocity, $\mathbf{v} = -2.5\mathbf{i} + 2\mathbf{j}$
The coefficients of **i** and **j** have the same ratio in both cases, but the signs of both are reversed: **u** and **v** are therefore antiparallel. The sphere is deflected through a total angle of 180°.

9 a First collision – let the angle the sphere makes with the wall after the collision be α, and its speed after impact v_1
For motion parallel to wall:
$v_1 \cos\alpha = 2.5\cos 45°$ **(1)**
For motion perpendicular to wall:
$v_1 \sin\alpha = 0.6 \times 2.5 \sin 45°$ **(2)**

Dividing equation **(2)** by equation **(1)** gives:
$\tan\alpha = 0.6\tan 45°$

$\tan\alpha = 0.6 \Rightarrow \cos\alpha = \frac{5}{\sqrt{34}}$

$\Rightarrow \alpha = 30.964\ldots = 31.0°$ (3 s.f.)
Substituting into equation **(1)** gives:

$v_1 \frac{5}{\sqrt{34}} = \frac{5}{2\sqrt{2}}$

$v_1 = \frac{\sqrt{17}}{2} = 2.0615... = 2.06\,\text{m s}^{-1}$

9 b First collision – let the angle the sphere makes with the wall after the collision be β, and its speed after impact v_2

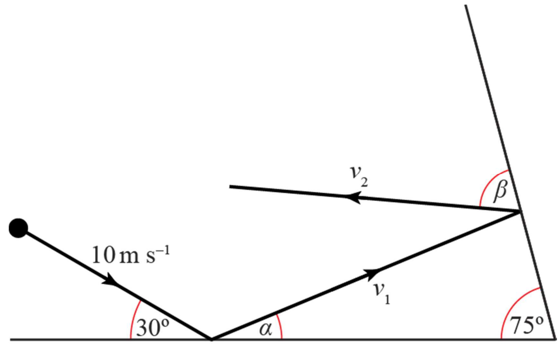
For motion parallel to wall:
$v_2 \cos\beta = v_1 \cos(60° - \alpha)$ **(3)**

For motion perpendicular to wall:
$v_2 \sin\beta = 0.6 v_1 \sin(60° - \alpha)$ **(4)**

Dividing equation **(4)** by equation **(3)** gives:
$\tan\beta = 0.6 \times \tan(60° - 30.964°) = 0.33308...$
$\Rightarrow \beta = 18.421°$
$\Rightarrow \cos\beta = 0.9488$

Substituting into equation **(3)** gives:
$$v_2 \times 0.9488 = \frac{\sqrt{17}}{2} \times 0.87431 = 1.8997 \text{ (4 d.p.)}$$
$$\text{Kinetic energy after second collision} = \frac{1}{2} \times 0.1 \times 1.8997^2 = 0.180 \text{ J (3 s.f.)}$$

10



First collision

For motion parallel to wall:
$v_1\cos\alpha = 10\cos 30°$ **(1)**

For motion perpendicular to wall:
$v_1\sin\alpha = 0.7\times 10\sin 30°$ **(2)**

Dividing equation **(2)** by equation **(1)** gives:
$\tan\alpha = 0.7\tan 30°$

$\tan\alpha = \frac{7}{10\sqrt{3}} \Rightarrow \cos\alpha = \frac{10\sqrt{3}}{\sqrt{349}}$

$\Rightarrow \alpha = 22.005°$ (3 d.p.)

Substituting into equation **(1)** gives:
$v_1\frac{10\sqrt{3}}{\sqrt{349}} = 10\frac{\sqrt{3}}{2} = \frac{\sqrt{349}}{2}$

First collision

For motion parallel to wall:
$v_2\cos\beta = v_1\cos(180° - 75° - \alpha) = v_1\cos 82.994°$ **(3)**

For motion perpendicular to wall:
$v_2\sin\beta = 0.5v_1\sin(180° - 75° - \alpha) = 0.5v_1\sin 82.994°$ **(4)**

Dividing equation **(4)** by equation **(3)** gives:
$\tan\beta = 0.5\tan 82.994° = 4.0687$

$\beta = 76.192°$ (3 d.p.)

Substituting into equation **(3)** gives:

$v_2\cos 76.192° = \frac{\sqrt{349}}{2}\cos 82.994° = 4.7736$

$$\begin{aligned}\text{Loss of kinetic energy} &= \frac{1}{2}m(u^2 - v_2^{\,2})\\ &= \frac{1}{2}\times 2\left(10^2 - 4.7736^2\right)\\ &= 77.2\text{ J (3 s.f.)}\end{aligned}$$

Challenge

a i Vertical component of motion is not changed by collision with wall. Therefore time taken for ball to reach floor for the first time is time taken for ball to fall 1 m from rest.

$$s = ut + \frac{1}{2}at^2$$

$$1 = \frac{1}{2} \times 9.8 \times t^2$$

$$t = \frac{1}{\sqrt{4.9}} = 0.45175...$$

Horizontal component of motion is 20 m s^{-1} before collides with wall and 0.6 × 20 = 12 m s^{-1} after collision. So time taken to reach wall:

$$t_1 = \frac{s}{v} = \frac{2.4}{20} = 0.12 \text{ s}$$

So time for which ball travelling at 12 m s^{-1} is t_2 = 0.45175 − 0.12 = 0.331 75

Distance travelled in this time = vt_2 = 12 × 0.33175 = 3.98 m (3 s.f.)

The ball first bounces on the floor 3.98 m from the wall.

ii Vertical component of speed immediately before it hits the floor:

$$v^2 = u^2 + 2as$$

$$v^2 = 2 \times 9.8 \times 1$$

$$v = \sqrt{19.6}$$

Vertical component of speed immediately after it hits the floor = $0.6\sqrt{19.6}$

Maximum height given when vertical component is again zero, so:

$$v^2 = u^2 + 2as$$

$$0 = (0.6^2 \times 19.6) - (2 \times 9.8 \times s)$$

$$\Rightarrow s = 0.6^2 = 0.36 \text{ m}$$

The maximum height after this bounce is 0.36 m.

b If the ball is hotter and e increases, the horizontal speed after hitting the wall will be greater and the ball will therefore travel further before hitting the floor.

Similarly, the vertical component of the speed immediately after it hits the floor will be greater and the maximum height after the first bounce will also be greater.