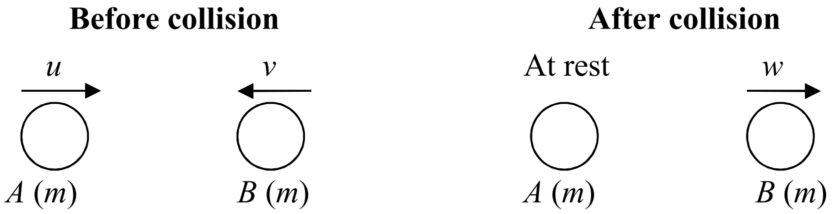
Elastic collisions in one dimension Mixed Exercise 4

1



Using conservation of linear momentum for the system $(\rightarrow)$:

$mu - mv = mw$

$\Rightarrow u - v = w$ **(1)**

Using Newton's law of restitution:

$e = \frac{1}{3} = \frac{w}{u-(-v)}$

$\Rightarrow u + v = 3w$ **(2)**

Adding equations **(1)** and **(2)** gives:

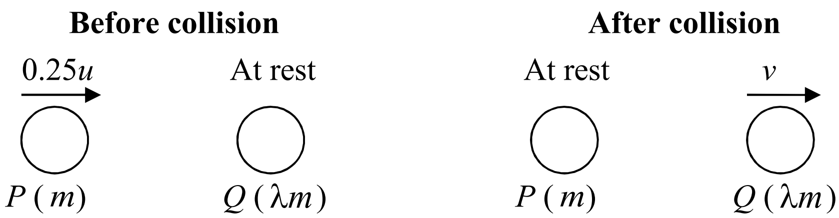
$2u = 4w \Rightarrow u = 2w$

Substituting in equation **(2)** gives:

$2w + v = 3w \Rightarrow v = w$

The ratio of the speeds before impact is $u : v = 2w : w = 2 : 1$ as required.

2



Using conservation of linear momentum for the system $(\rightarrow)$:

$0.25mu = \lambda mv \Rightarrow u = 4\lambda v$ **(1)**

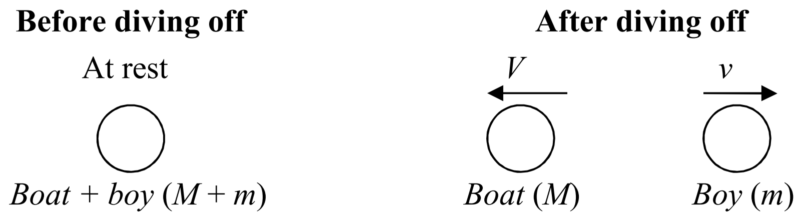
Using Newton's law of restitution:

$e = \frac{1}{4} = \frac{v}{0.25u} \Rightarrow u = 16v$ **(2)**

From equations **(1)** and **(2)**:

$u = 16v = 4\lambda v \Rightarrow \lambda = 4$

3 a Note that the boat moves in the opposite direction to the boy after the boy dives off.



Using conservation of linear momentum for the system $(\rightarrow)$:

$$0 = mv - MV$$

$$\Rightarrow V = \frac{mv}{M}$$

b Let total kinetic energy of boy and boat after the dive be KE

$$KE = \frac{1}{2}MV^2 + \frac{1}{2}mv^2$$

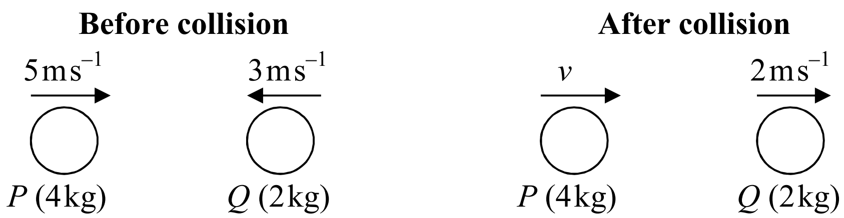
$$= \frac{1}{2}M\left(\frac{mv}{M}\right)^2 + \frac{1}{2}mv^2$$

$$= \frac{m^2v^2 + mMv^2}{2M}$$

$$= \frac{m(m+M)v^2}{2M} \qquad \text{as required}$$

c The boat is large and heavy, so there will be additional tilting/rolling motion. The boat is also on water, so given waves, tides and currents it is unlikely to be at rest initially.

4



Using conservation of linear momentum for the system $(\rightarrow)$:

$$4 \times 5 + 2 \times (-3) = 4v + 2 \times 2$$

$$4v = 10 \Rightarrow v = 2.5\,\text{m s}^{-1}$$

Loss of kinetic energy = initial kinetic energy − final kinetic energy

$$= \frac{1}{2} \times 4 \times 5^2 + \frac{1}{2} \times 2 \times 3^2 - \left(\frac{1}{2} \times 4 \times 2.5^2 + \frac{1}{2} \times 2 \times 2^2\right)$$

$$= 50 + 9 - 12.5 - 4 = 42.5\ \text{J}$$

5 a

Using conservation of linear momentum for the system $(\rightarrow)$:

$$3mu = 3mv + mw$$
$$\Rightarrow 3v + w = 3u \qquad \textbf{(1)}$$

Using Newton's law of restitution:

$$e = \frac{w - v}{u}$$
$$\Rightarrow w - v = eu \qquad \textbf{(2)}$$

Subtracting equation **(2)** from equation **(1)** gives:

$$4v = 3u - eu \Rightarrow v = \frac{u(3-e)}{4}$$

b Substituting for v in equation **(2)** gives:

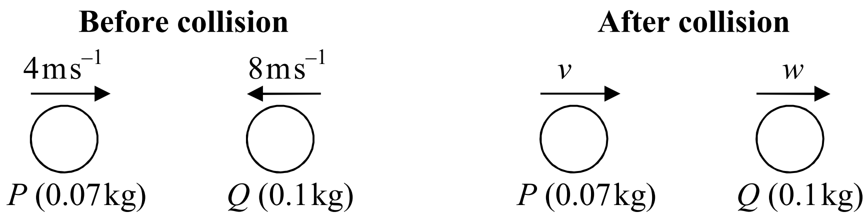
$$w = eu + \frac{u(3-e)}{4} = \frac{4eu + 3u - eu}{4} = \frac{3u(e+1)}{4}$$

Loss of kinetic energy = initial kinetic energy − final kinetic energy

$$= \frac{1}{2} \times 3mu^2 - \frac{1}{2} \times 3mv^2 - \frac{1}{2}mw^2$$
$$= \frac{m}{2}\left(3u^2 - 3\frac{u^2(3-e)^2}{16} - 9\frac{u^2(1+e)^2}{16}\right)$$
$$= \frac{3mu^2}{32}\left(16 - (9 - 6e + e^2) - (3 + 6e + 3e^2)\right)$$
$$= \frac{3mu^2}{32}(4 - 4e^2)$$
$$= \frac{3}{8}mu^2(1 - e^2)$$

c Impulse exerted on Q is change of momentum of $Q = mw = \dfrac{3mu(1+e)}{4}$ N s

6 a



Using conservation of linear momentum for the system $(\rightarrow)$:

$0.07\times 4+0.1\times(-8)=0.07v+0.1w$

$\Rightarrow\ 7v+10w=-52$ **(1)**

Using Newton's law of restitution:

$e=\dfrac{5}{12}=\dfrac{w-v}{4-(-8)}$

$\Rightarrow w-v=5$ **(2)**

Adding equation **(1)** and 7 × equation **(2)** gives:

$17w=-52+35=-17\Rightarrow w=-1$

Substituting in equation **(2)** gives:

$-1-v=5\Rightarrow v=-6$

So the velocities after impact are $6\,\text{m s}^{-1}$ and $1\,\text{m s}^{-1}$ in the direction of the 100 g mass prior to the impact.

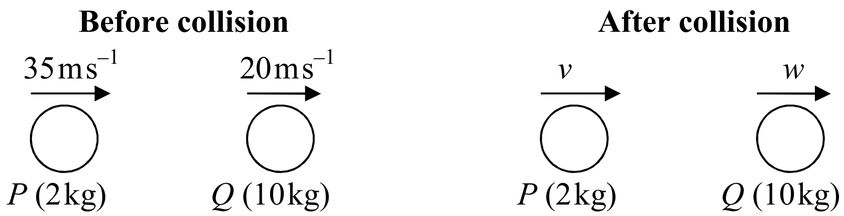
b Let loss of kinetic energy in the collision be KE

KE = initial kinetic energy – final kinetic energy

$=\dfrac{1}{2}\times 0.07\times 4^2+\dfrac{1}{2}\times 0.1\times(-8)^2-\left(\dfrac{1}{2}\times 0.07\times(-6)^2+\dfrac{1}{2}\times 0.1\times(-1)^2\right)$

$=(0.56+3.2)-(1.26+0.05)=2.45\text{ J}$

7



Using conservation of linear momentum for the system $(\rightarrow)$:

$2\times35+10\times20=2v+10w$

$\Rightarrow 2v+10w=270$ **(1)**

Using Newton's law of restitution:

$e=\frac{3}{5}=\frac{w-v}{35-20}$

$\Rightarrow w-v=9$ **(2)**

Adding equation **(1)** and 2 × equation **(2)** gives:

$12w=270+18=288\Rightarrow w=24$

Substituting in equation **(2)** gives:

$24-v=9\Rightarrow v=15$

After the impact, assume that the particles move at constant speed and use speed × time = distance.

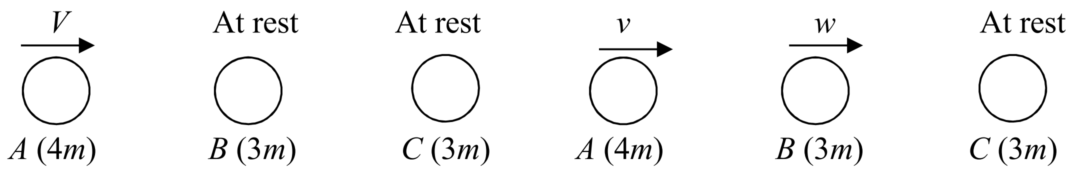
Five seconds after the impact the 10kg mass moved a distance $24\times5=120\,\text{m}$

It takes the 2kg mass a time of $\frac{120}{15}$ to travel 120 m, i.e. 8 seconds.

The time that elapses between the 10kg sphere resting on the barrier and it being struck by the 2 kg sphere therefore $=8\,\text{s}-5\,\text{s}=3$ seconds

8 First consider impact of A with B, then of B with C, then of A with B again.

Before the first collision **After the first collision**



Using conservation of linear momentum for the system $(\rightarrow)$::

$4V = 4v + 3w \quad \Rightarrow 4v + 3w = 4V$ **(1)**

Using Newton's law of restitution:

$e = \frac{3}{4} = \frac{w - v}{V} \quad \Rightarrow 4w - 4v = 3V$ **(2)**

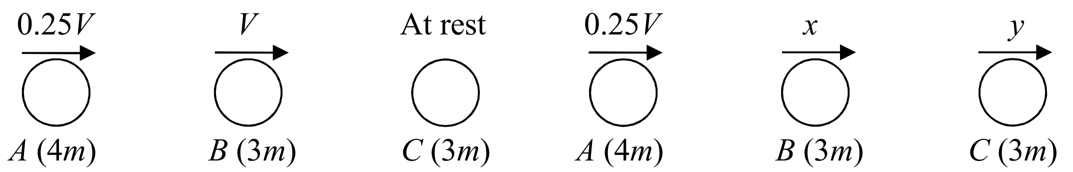
Adding equations **(1)** and **(2)** gives:

$7w = 7V \Rightarrow w = V$

Substituting in equation **(2)** gives:

$4V - 4v = 3V \Rightarrow v = 0.25V$

Before the second collision **After the second collision**



Using conservation of linear momentum for the system $(\rightarrow)$::

$3V = 3x + 3y \quad \Rightarrow x + y = V$ **(3)**

Using Newton's law of restitution:

$e = \frac{3}{4} = \frac{y - x}{V} \quad \Rightarrow \ y - x = 0.75V$ **(4)**

Adding equations **(3)** and **(4)** gives:

$2y = 1.75V \Rightarrow y = 0.875V$

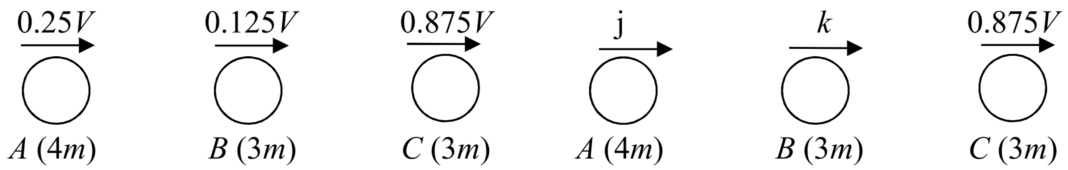
Substituting in equation **(4)** gives:

$0.875V - x = 0.75V \Rightarrow x = 0.125V$

Ball A is now moving at $0.25V$ and ball B is moving at $0.125V$ so ball A will strike ball B for a second time.

8 continued

Before the third collision **After the third collision**



Using conservation of linear momentum for the system (→)::

$(4\times0.25)V+(3\times0.125)V=4j+3k$

$\Rightarrow 4j+3k=1.375V$ **(5)**

Using Newton's law of restitution:

$e=\frac{3}{4}=\frac{k-j}{0.125V}$

$\Rightarrow\ 4k-4j=0.375V$ **(6)**

Adding equations **(5)** and **(6)** gives:

$7k=1.75V\Rightarrow k=0.25V$

Substituting in equation **(6)** gives:

$V-4j=0.375V\Rightarrow j=0.15625V$

After three collisions the velocities are $0.15625V$, $0.25V$ and $0.875V$ for balls A, B and C respectively.

In fractions, the respective velocities are $\frac{5}{32}V, \frac{1}{4}V$ and $\frac{7}{8}V$.

As $\frac{5}{32}V<\frac{1}{4}V<\frac{7}{8}V$ there are no further collisions.

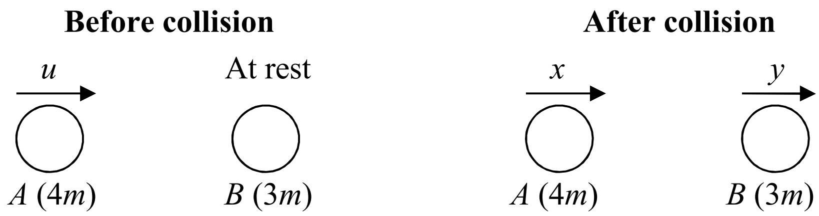
9 a Velocity of bullet after hitting the barrier $=600\times0.4=240\text{ m s}^{-1}$

Kinetic energy lost $=\frac{1}{2}\times0.06\times600^2-\frac{1}{2}\times0.06\times240^2$

$=9072\text{ J}$

b *Either* heat *or* sound.

10 a



Using conservation of linear momentum for the system $(\rightarrow)$:

$4u = 3y + 4x$

$\Rightarrow 3y + 4x = 4u$ **(1)**

Using Newton's law of restitution:

$e = \dfrac{y - x}{u}$

$\Rightarrow y - x = eu$ **(2)**

Adding equation **(1)** and 4 × equation **(2)** gives:

$7y = 4u + 4eu \Rightarrow y = \dfrac{4}{7}u(1+e)$

Substituting in equation **(2)** gives:

$\dfrac{4}{7}u(1+e) - x = eu$

$\Rightarrow x = \dfrac{4u + 4eu - 7eu}{7} = \dfrac{u}{7}(4 - 3e)$

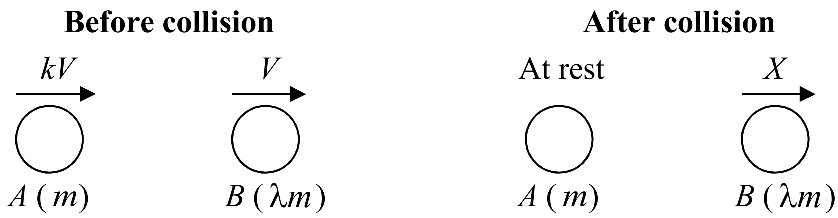
b Impulse = change in momentum of B

So $2mu = 3m \times \dfrac{4}{7}u(1+e)$

$1 + e = \dfrac{14}{12}$

$\Rightarrow e = \dfrac{1}{6}$

11 a



Using conservation of linear momentum for the system $(\rightarrow)$:

$mkV + \lambda mV = \lambda mX$

$\Rightarrow \; X = \dfrac{(\lambda + k)V}{\lambda}$

Using Newton's law of restitution:

$$e = \frac{X}{kV - V}$$

$$= \frac{(\lambda + k)V}{\lambda(kV - V)} \qquad \text{(substituting for } X\text{)}$$

$$= \frac{\lambda + k}{\lambda(k-1)}$$

b As $e < 1$, $\dfrac{\lambda + k}{\lambda(k-1)} < 1$

So $\lambda + k < \lambda k - \lambda$ (as $\lambda > 0$ and $k > 1$)

$2\lambda + k < \lambda k$

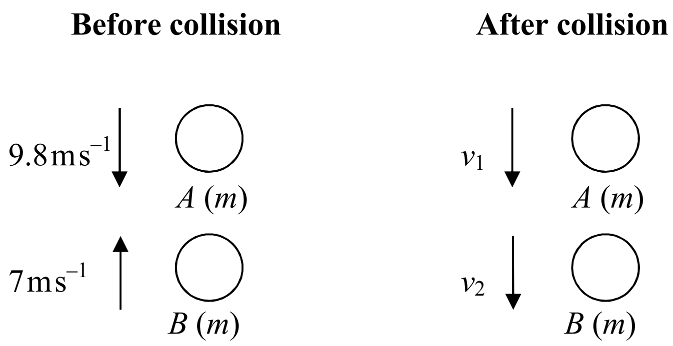
$\lambda k - 2\lambda > k$

$\lambda(k-2) > k$

Since $k > 0$ and $\lambda > 0$, therefore $k - 2 > 0$

So $\lambda > \dfrac{k}{k-2}$ and $k > 2$

12 a Use $v = u + at$ downwards with $u = 0, t = 1$ and $a = g = 9.8$ to find the velocity of the first ball before impact. This gives:
$v = 9.8$



Using conservation of linear momentum for the system $(\downarrow)$:

$9.8m - 7m = mv_2 + mv_1$

$\Rightarrow v_2 + v_1 = 2.8$ **(1)**

Using Newton's law of restitution:

$e = \frac{1}{4} = \frac{v_2 - v_1}{9.8 + 7}$

$\Rightarrow v_2 - v_1 = 4.2$ **(2)**

Adding equations **(1)** and **(2)** gives:

$2v_2 = 7 \Rightarrow v_2 = 3.5\,\text{m s}^{-1}$

Substituting in equation **(2)** gives:

$3.5 - v_1 = 4.2$

$\Rightarrow v_1 = -0.7\,\text{m s}^{-1}$

Both balls change directions, the first moves up with speed $0.7\,\text{m s}^{-1}$ and the second moves down with speed $3.5\,\text{m s}^{-1}$.

b Kinetic energy before impact $= \frac{1}{2}m \times 9.8^2 + \frac{1}{2}m \times 7^2 = 72.52m$ J

Kinetic energy after impact $= \frac{1}{2}m \times 0.7^2 + \frac{1}{2}m \times 3.5^2 = 6.37m$ J

Percentage loss of kinetic energy $= \frac{72.52 - 6.37}{72.52} = 91.2\% = 91\%$ (2 s.f.)

13 a Stage one: particle falls under gravity ↓:

Use $v^2 = u^2 + 2as$ downwards with $u = 0, s = 8$ and $a = g$

$v^2 = 2g \times 8 = 16g \Rightarrow v = \sqrt{16g}$

Stage two: first impact:

The particle rebounds with velocity $\frac{1}{4}\sqrt{16g} = \sqrt{g}$

Stage three: particle moves under gravity ↑:

Let the height to which the ball rebounds after the first bounce be h_1

Use $v^2 = u^2 + 2as$ upwards with $v = 0, u = \sqrt{g}, a = -g$ and $s = h_1$

$0 = g - 2gh_1$

$\Rightarrow h_1 = 0.5\text{m}$

b Use $v = u + at$ upwards with $v = 0, u = \frac{1}{4}\sqrt{16g}$ and $a = -g$ to find the time it takes the particle to reach the top of the bounce

$0 = \frac{1}{4}\sqrt{16g} - gt$

$\Rightarrow t = \frac{\sqrt{g}}{g} = 0.319$

So the time taken to reach the plane again $= 2 \times 0.319 = 0.64$ s (2 s.f.) or $\frac{2}{\sqrt{g}}$ s

c Speed of approach $= \sqrt{g}$

The speed of the particle after the second rebound $= e\sqrt{g} = \frac{\sqrt{g}}{4} = 0.78\,\text{m}\,\text{s}^{-1}$ (2 s.f.)

14 Stage one: particle falls under gravity $\downarrow$:

Use $v^2 = u^2 + 2as$ downwards with $u = 0, s = h$ and $a = g$

$v^2 = 2gh \Rightarrow v = \sqrt{2gh}$

Use $s = ut + \frac{1}{2}at^2$ to find the time to the first bounce

$h = \frac{1}{2}gt_1^2 \Rightarrow t_1 = \sqrt{\frac{2h}{g}}$

Stage two: particle rebounds from plane.

The particle rebounds with velocity $e\sqrt{2gh}$

Stage three: particle moves under gravity until it hits the plane again $\uparrow$:

Use $s = ut + \frac{1}{2}at^2$ to find the time from the first to the second bounce, $u = e\sqrt{2gh}$, $s = 0$ and $a = -g$

$0 = e\sqrt{2gh}t_2 - \frac{1}{2}gt_2^2$

$t_2 = \frac{2e\sqrt{2gh}}{g} = 2e\sqrt{\frac{2h}{g}}$

Stage four: particle rebounds (again) from plane.

Speed of approach $= e\sqrt{2gh}$, so speed of rebound $= e^2\sqrt{2gh}$

Similar working finds that the time from the second bounce to the third bounce is $t_3 = 2e^2\sqrt{\frac{2h}{g}}$

And the time from the third bounce to the fourth bounce is $t_4 = 2e^3\sqrt{\frac{2h}{g}} \ldots$

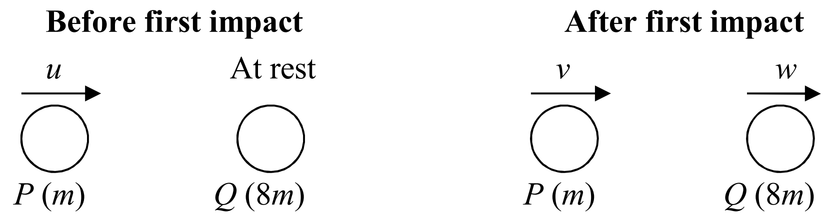
Let the total time taken by the particle be T, then

$$T = \sqrt{\frac{2h}{g}} + 2e\sqrt{\frac{2h}{g}} + 2e^2\sqrt{\frac{2h}{g}} + 2e^3\sqrt{\frac{2h}{g}} + \ldots$$
$$= \sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2h}{g}}(e + e^2 + e^3 + \ldots)$$

The expression in the bracket is an infinite geometric series with $a = e$ and $r = e$. Using the formula $S_\infty = \frac{a}{1-r} = \frac{e}{1-e}$, the expression for T can be simplified as follows

$$T = \sqrt{\frac{2g}{h}}\left(1 + \frac{2e}{1-e}\right) = \left(\frac{1-e+2e}{1-e}\right)\sqrt{\frac{2h}{g}} = \frac{1+e}{1-e}\sqrt{\frac{2h}{g}}$$

15



Using conservation of linear momentum for the system $(\rightarrow)$:

$$mu = mv + 8mw \quad \Rightarrow v + 8w = u \qquad \textbf{(1)}$$

Using Newton's law of restitution:

$$e = \frac{7}{8} = \frac{w - v}{u} \quad \Rightarrow 8w - 8v = 7u \qquad \textbf{(2)}$$

Subtracting equation **(2)** from equation **(1)** gives:

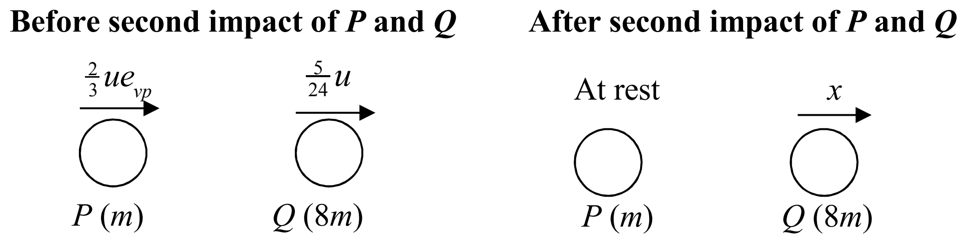
$$9v = u - 7u \Rightarrow v = -\frac{2}{3}u$$

Substituting in equation **(2)** gives:

$$8w + \frac{16u}{3} = 7u \Rightarrow 8w = \frac{5u}{3} \Rightarrow w = \frac{5u}{24}$$

Let e_{vp} be the coefficient of restitution between P and the vertical place.

So P then hits the vertical plane with speed $\frac{2u}{3}$ and rebounds with speed $\frac{2}{3}ue_{vp}$



Using conservation of linear momentum for the system $(\rightarrow)$:

$$\frac{2}{3}mue_{vp} + \frac{5}{3}mu = 8mx \quad \Rightarrow 24x = 2ue_{vp} + 5u \qquad \textbf{(1)}$$

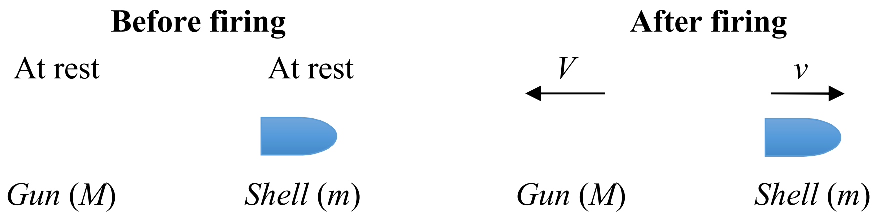
Using Newton's law of restitution:

$$e = \frac{7}{8} = \frac{x}{-} \quad \Rightarrow \frac{7}{8}\left(\frac{2}{3}ue_{vp} - \frac{5}{24}u\right) = x \quad \Rightarrow 24x = 14ue_{vp} - \frac{35}{8}u \qquad \textbf{(2)}$$

Subtracting equation **(2)** from equation **(1)** gives:

$$12ue_{vp} = 5u + \frac{35}{8}u = \frac{75}{8}u \quad \Rightarrow e_{vp} = \frac{75}{96} = \frac{25}{32}$$

16



Using conservation of linear momentum for the system (→):

$mv - MV = 0 \quad \Rightarrow V = \dfrac{mv}{M}$ **(1)**

Energy released: $E = \dfrac{1}{2}mv^2 + \dfrac{1}{2}MV^2$ **(2)**

Substituting for V into equation **(2)** gives:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}M\frac{m^2v^2}{M^2}$$

$$2ME = mMv^2 + m^2v^2$$

$$v^2 = \frac{2ME}{m(M+m)}$$

$$\Rightarrow v = \sqrt{\frac{2ME}{m(M+m)}}\ \text{m s}^{-1}$$

17 a Using $v^2 = u^2 + 2as$ downwards with $u = 0, s = H$ and $a = g$

$v^2 = 2gH \quad \Rightarrow v = \sqrt{2gH}$

The ball rebounds with speed $e\sqrt{2gH}$

Using $v^2 = u^2 + 2as$ upwards with $u = e\sqrt{2gH}, s = h$ and $a = -g$

$0 = 2gHe^2 - 2gh$

$e^2 = \dfrac{h}{H} \quad \Rightarrow e = \sqrt{\dfrac{h}{H}}$

b The ball rebounds the second time with speed $e^2\sqrt{2gH}$

Using $v^2 = u^2 + 2as$ upwards with $u = e^2\sqrt{2gH}, s = h'$ and $a = -g$

$0 = 2gHe^4 - 2gh'$

$$h' = He^4 = H\left(\frac{h}{H}\right)^2 = \frac{Hh^2}{H^2} = \frac{h^2}{H}$$

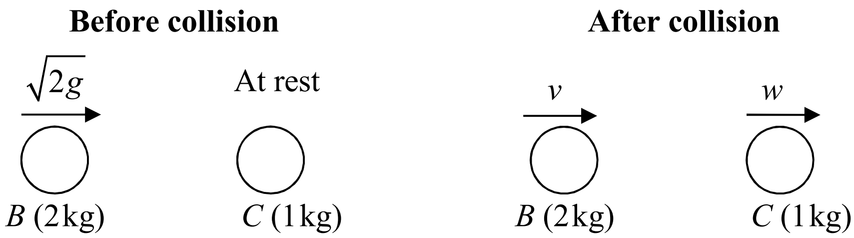
$$h' = e^4H = \left(\frac{h}{H}\right)^2 H = \frac{h^2H}{H^2} = \frac{h^2}{H}$$

c The ball continues to bounce (for an infinite amount of time) with its height decreasing by a common ratio each time.

18 a Use $F = ma$ to determine the acceleration of the sphere down the smooth slope. This gives:

$$2g\sin 30° = 2a \Rightarrow a = g\sin 30° = \frac{g}{2}$$

Use $v^2 = u^2 + 2as$ with $u = 0, s = 2$ and $a = 0.5g$ to find the speed of the ball when it reaches the horizontal plane: $v^2 = 2g \Rightarrow v = \sqrt{2g}$



Using conservation of linear momentum for the system $(\rightarrow)$::

$$2\sqrt{2g} = 2v + w$$

$$\Rightarrow 2v + w = 2\sqrt{2g} \qquad \textbf{(1)}$$

Using Newton's law of restitution:

$$e = 0.75 = \frac{w - v}{\sqrt{2g}}$$

$$\Rightarrow w - v = 0.75\sqrt{2g} \qquad \textbf{(2)}$$

Adding equation **(1)** and 2 × equation **(2)** gives:

$$3w = 2\sqrt{2g} + 1.5\sqrt{2g} = 3.5\sqrt{2g} \Rightarrow w = \frac{7}{6}\sqrt{2g}\ \text{m s}^{-1}$$

Substituting in equation **(2)** gives:

$$\frac{7}{6}\sqrt{2g} - v = \frac{3}{4}\sqrt{2g}$$

$$\Rightarrow v = \left(\frac{14}{12} - \frac{9}{12}\right)\sqrt{2g} = \frac{5}{12}\sqrt{2g}\ \text{m s}^{-1}$$

Both B and C continue in the direction B was originally moving.

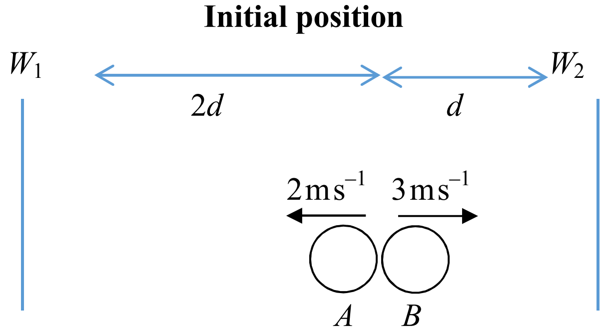
b Energy lost in the collision = initial kinetic energy – final kinetic energy

$$= \frac{1}{2}\times 2\times\left(\sqrt{2g}\right)^2 - \left(\frac{1}{2}\times 2\times\left(\frac{5\sqrt{2g}}{12}\right)^2 + \frac{1}{2}\times 1\times\left(\frac{7\sqrt{2g}}{6}\right)^2\right)$$

$$= 2g - \left(\frac{50g}{144} + \frac{98g}{72}\right) = 2g - \left(\frac{50g}{144} + \frac{98g}{72}\right) = \frac{42g}{144} = \frac{7g}{24}\ \text{J}$$

c If $e < 0.75$ the amount of kinetic energy lost would increase as the collision would be less elastic.

19



Suppose point Q is at a distance x from wall W_1

Consider the motion of sphere A:

Time taken for A to travel from point P to wall W_1 is $\frac{\text{distance}}{\text{speed}} = \frac{2d}{2} = d$

Sphere A rebounds with speed $\frac{3}{5} \times 2 = \frac{6}{5}\,\text{m s}^{-1}$

Time taken for A to travel from wall W_1 to point Q is $\frac{\text{distance}}{\text{speed}} = \frac{x}{\frac{6}{5}} = \frac{5x}{6}$

Consider the motion of sphere B:

Time taken for B to travel from point P to wall W_2 is $\frac{\text{distance}}{\text{speed}} = \frac{d}{3}$

Sphere B rebounds with speed $\frac{3}{5} \times 3 = \frac{9}{5}\,\text{m s}^{-1}$

Time taken for B to travel from W_2 to point Q is $\frac{\text{distance}}{\text{speed}} = \frac{3d - x}{\frac{9}{5}} = \frac{5(3d - x)}{9} = \frac{15d - 5x}{9}$

When A and B meet at Q, they have been travelling for the same time, so

$$d + \frac{5x}{6} = \frac{d}{3} + \frac{15d - 5x}{9}$$

$$18d + 15x = 6d + 30d - 10x$$

$$25x = 18d$$

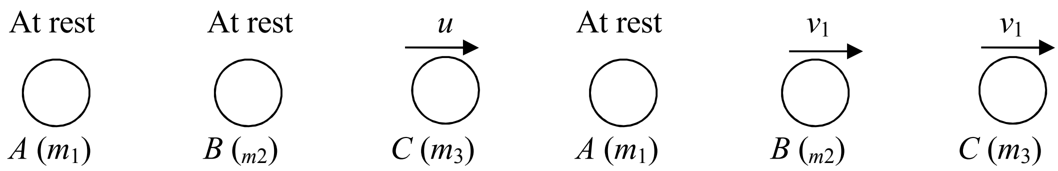
$$\Rightarrow x = \frac{18d}{25} \text{ and } 3d - x = \frac{57d}{25}$$

$$18d + 15x = 6d + 30d - 10x$$

Therefore the distance ratio $W_1Q : W_2Q = x : 3d - x = \frac{18d}{25} : \frac{57d}{25} = 18 : 57 = 6 : 19$

Challenge

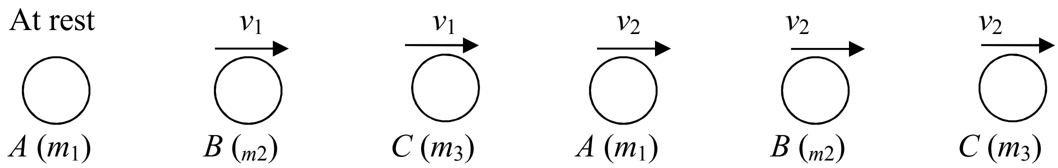
Before string B–C becomes taut **After string B–C becomes taut**



Using conservation of linear momentum for the system $(\rightarrow)$:

$$m_3u = m_2v_1 + m_3v_1$$
$$m_3u = v_1(m_2 + m_3)$$
$$\Rightarrow v_1 = \frac{m_3u}{(m_2 + m_3)}$$

Before string A–B becomes taut **After string A–B becomes taut**



Using conservation of linear momentum for the system $(\rightarrow)$:

$$m_2v_1 + m_3v_1 = m_1v_2 + m_2v_2 + m_3v_2$$
$$v_1(m_2 + m_3) = v_2(m_1 + m_2 + m_3)$$
$$\Rightarrow v_2 = \frac{v_1(m_2 + m_3)}{(m_1 + m_2 + m_3)} = \frac{m_3u}{(m_1 + m_2 + m_3)}$$

$$\text{Total kinetic energy} = \frac{1}{2}(m_1 + m_2 + m_3)v_2^2$$
$$= \frac{1}{2}(m_1 + m_2 + m_3)\left(\frac{m_3u}{(m_1 + m_2 + m_3)}\right)^2$$
$$= \frac{m_3^2u^2}{2(m_1 + m_2 + m_3)}$$