Elastic collisions in one dimension 4D

1 a First collision (between A and B)

Using conservation of linear momentum for the system $(\rightarrow)$:

$2\times5+1\times1=2u+v$

$\Rightarrow 2u+v=11$ **(1)**

Using Newton's law of restitution:

$e=\frac{1}{2}=\frac{v-u}{5-1}$

$\Rightarrow v-u=2$ **(2)**

Subtracting equation **(2)** from equation **(1)** gives:

$3u=9\Rightarrow u=3$

Substituting into equation **(1)** gives:

$6+v=11\Rightarrow v=5$ [This result can be checked in equation **(2)**]

Second collision (between B and C)

Using conservation of linear momentum for the system $(\rightarrow)$:

$1\times5+2\times4=x+2y$

$\Rightarrow x+2y=13$ **(3)**

Using Newton's law of restitution:

$e=\frac{1}{2}=\frac{y-x}{5-4}$

$\Rightarrow y-x=\frac{1}{2}$ **(4)**

Adding equations **(3)** and **(4)** gives:

$3y=\frac{27}{2}\Rightarrow y=\frac{9}{2}=4.5$

Substituting into equation **(4)** gives:

$4.5-x=\frac{1}{2}\Rightarrow x=4$

Solution: $u=3,\ v=5,\ x=4,\ y=4.5$

1 b First collision (between A and B)
Using conservation of linear momentum for the system $(\rightarrow)$:

$1.5 \times 10 + 2 \times (-2) = 1.5u + 2v$

$\Rightarrow 1.5u + 2v = 11$ **(1)**

Using Newton's law of restitution:

$e = \frac{1}{6} = \frac{v-u}{10+2} = \frac{v-u}{12}$

$\Rightarrow v - u = 2$ **(2)**

Adding equation **(1)** to 1.5 × equation **(2)** gives:

$3.5v = 14 \Rightarrow v = 4$

Substituting into equation **(2)** gives:

$4 - u = 2 \Rightarrow u = 2$ [This result can be checked in equation **(1)**]

Second collision (between B and C)
Using conservation of linear momentum for the system $(\rightarrow)$:

$2 \times 4 + 1 \times 3 = 2x + y$

$\Rightarrow 2x + y = 11$ **(3)**

Using Newton's law of restitution:

$e = \frac{1}{2} = \frac{y-x}{4-3}$

$\Rightarrow y - x = \frac{1}{2}$ **(4)**

Subtracting equation **(4)** from equation **(3)** gives:

$3x = \frac{21}{2} \Rightarrow x = \frac{7}{2} = 3.5$

Substituting into equation **(4)** gives:

$y - 3.5 = \frac{1}{2} \Rightarrow y = 4$

Solution: $u = 2,\ v = 4,\ x = 3.5,\ y = 4$

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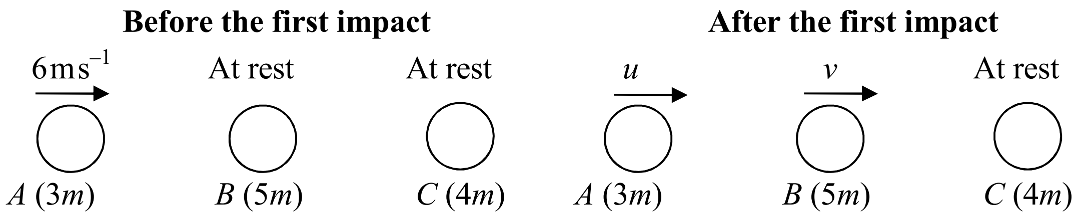


Using conservation of linear momentum for the system $(\rightarrow)$:

$3m \times 6 = 3mu + 5mv$

$\Rightarrow 3u + 5v = 18$ **(1)**

Perfectly elastic means e = 1. So using Newton's law of restitution:

$e = 1 = \dfrac{v - u}{6}$

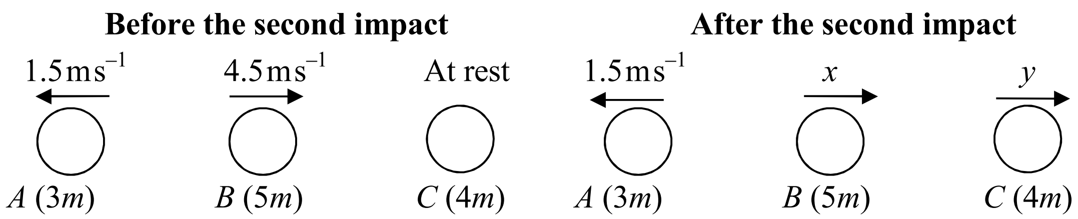
$\Rightarrow v - u = 6$ **(2)**

Adding equation **(1)** to 3 × equation **(2)** gives:

$8v = 36 \Rightarrow v = 4.5$

Substituting into equation **(2)** gives:

$4.5 - u = 6 \Rightarrow u = -1.5$



Using conservation of linear momentum for the system $(\rightarrow)$:

$5m \times 4.5 = 5mx + 4my$

$\Rightarrow 5x + 4y = 22.5$ **(3)**

Using Newton's law of restitution:

$e = 1 = \dfrac{y - x}{4.5}$

$\Rightarrow y - x = 4.5$ **(4)**

Adding equation **(3)** to 5 × equation **(4)** gives:

$9y = 22.5 + 22.5 = 45 \Rightarrow y = 5$

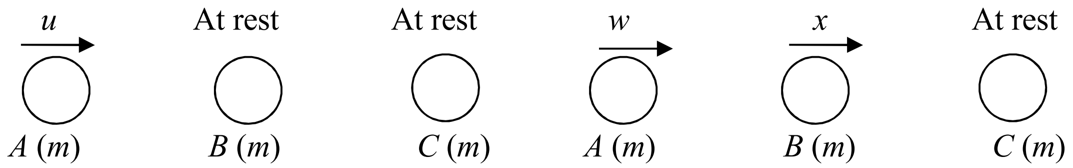
Substituting into equation **(4)** gives:

$5 - x = 4.5 \Rightarrow x = 0.5$

Solution: $A\ -1.5\,\text{m s}^{-1}$, $B\ 0.5\,\text{m s}^{-1}$, $C\ 5\,\text{m s}^{-1}$

3 a

Before the first impact **After the first impact**



Using conservation of linear momentum for the system $(\rightarrow)$:

$u = w + x \Rightarrow w + x = u$ **(1)**

Using Newton's law of restitution:

$e = \dfrac{x - w}{u}$

$\Rightarrow x - w = eu$ **(2)**

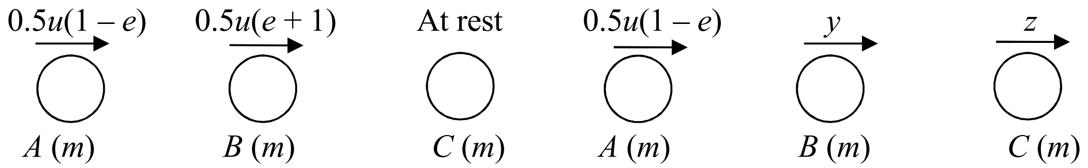
Adding equation **(1)** to equation **(2)** gives:

$2x = eu + u \Rightarrow x = 0.5u(e+1)$

Substituting into equation **(2)** gives:

$0.5u(e+1) - w = eu \Rightarrow w = 0.5u(1-e)$

Before the second impact **After the second impact**



Using conservation of linear momentum for the system $(\rightarrow)$:

$0.5u(e+1) = y + z$

$\Rightarrow y + z = 0.5u(e+1)$ **(3)**

Using Newton's law of restitution:

$e = \dfrac{z - y}{0.5u(e+1)}$

$\Rightarrow z - y = 0.5ue(e+1)$ **(4)**

Adding equation **(3)** to equation **(4)** gives:

$2z = 0.5u(e+1) + 0.5eu(e+1)$

$\Rightarrow z = 0.25u(e+1)(1+e) = 0.25u(1+e)^2$

Substituting into equation **(4)** gives:

$0.25u(1+e)^2 - y = 0.5eu(e+1)$

$\Rightarrow y = u(1+e)(0.25 + 0.25e - 0.5e) = 0.25u(1+e)(1-e)$

Solution: $0.5u(1-e),\ 0.25u(1+e)(1-e),\ 0.25u(1+e)^2$

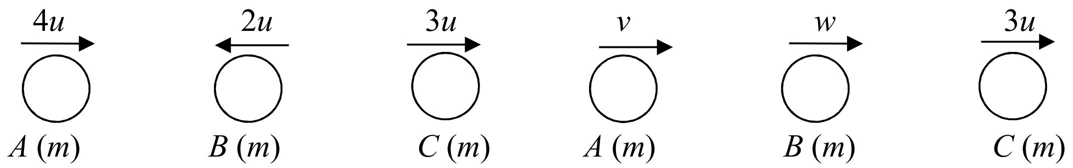
3 b A will catch up with B provided that

$0.5u(1-e) > 0.25u(1+e)(1-e)$, i.e. provided that $2 > 1+e$

Since $e < 1$ this condition holds and A will catch up with B resulting in a further collision.

4 a

Before the first collision **After the first collision**



Using conservation of linear momentum for the system $(\rightarrow)$:

$4u - 2u = v + w$

$\Rightarrow v + w = 2u$ **(1)**

Using Newton's law of restitution:

$e = \dfrac{w-v}{4u+2u}$

$\Rightarrow w - v = 6eu$ **(2)**

Adding equations **(1)** and **(2)** gives:

$2w = 2u + 6ue$

$\Rightarrow w = u(1+3e)$

B will collide with C if the speed of B after collision with A is greater than the speed of C, i.e. if $w > 3u$. This will occur if:

$u(1+3e) > 3u$

$3e > 2 \Rightarrow e > \dfrac{2}{3}$

b Subtracting equation **(2)** from equation **(1)** gives:

$2v = 2u - 6eu$

$v = u(1-3e)$

If $e > \dfrac{2}{3}$ then $v < 0$, and the direction of A is reversed by the collision with B.

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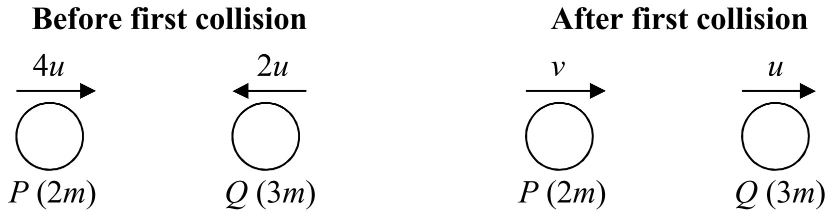


Using conservation of linear momentum for the system $(\rightarrow)$:

$2m\times 4u + 3m\times(-2u) = 2mv + 3mu$

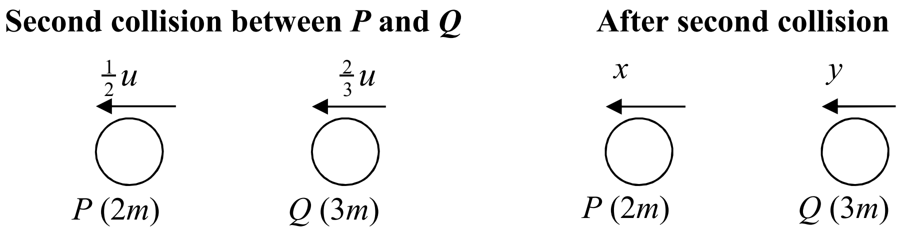
$2u = 2v + 3u \Rightarrow v = -0.5u$

Use this result to find the coefficient of restitution between particles P and Q.

$e = \dfrac{u - v}{4u + 2u} = \dfrac{1.5u}{6u} = 0.25$

The second collision is between Q and the wall.

Q rebounds from the wall with velocity $\dfrac{2}{3}u$, as the coefficient of restitution between Q and the wall is $\dfrac{2}{3}$.



Using conservation of linear momentum for the system $(\leftarrow)$:

$2m\times\dfrac{1}{2}u + 3m\times\dfrac{2}{3}u = 2x + 3y$

$\Rightarrow 2x + 3y = 3u \qquad (1)$

Using Newton's law of restitution:

$e = \dfrac{1}{4} = \dfrac{x - y}{\frac{2}{3}u - \frac{1}{2}u}$

$\Rightarrow x - y = \dfrac{1}{4}\times\dfrac{1}{6}u = \dfrac{u}{24} \qquad (2)$

Adding equation (1) to 3 × equation (2) gives:

$5x = 3u + \dfrac{u}{8} = \dfrac{25u}{8} \Rightarrow x = \dfrac{5u}{8}$

Substituting into equation (2) gives:

$\dfrac{5u}{8} - y = \dfrac{u}{24} \Rightarrow y = \dfrac{15u - u}{24} = \dfrac{14u}{24} = \dfrac{7u}{12}$

6 a

Before first collision **After first collision**

Using conservation of linear momentum for the system $(\rightarrow)$:

$m \times 12u = mv + 3mw$

$\Rightarrow v + 3w = 12u \qquad (1)$

Using Newton's law of restitution:

$e = \frac{2}{3} = \frac{w - v}{12u}$

$\Rightarrow w - v = 8u \qquad (2)$

Adding equation (1) to equation (2) gives:

$4w = 20u \Rightarrow w = 5u$

Substituting into equation (2) gives:

$5u - v = 8u \Rightarrow v = -3u$

After the collision the speed of P is $3u$ and its direction is reversed, and the speed of Q is $5u$.

b Q then hits a wall and rebounds with speed $\frac{4}{5} \times 5u = 4u$

Second collision between *P* and *Q* **After second collision**

Using conservation of linear momentum for the system $(\leftarrow)$:

$3mu + 12mu = mx + 3my \Rightarrow 3y + x = 15u \qquad$ **(3)**

Using Newton's law of restitution:

$e = \frac{2}{3} = \frac{x - y}{4u - 3u} \Rightarrow 3x - 3y = 2u \qquad$ **(4)**

Adding equation **(1)** to equation **(2)** gives:

$4x = 17u \Rightarrow x = \frac{17u}{4}$

Substituting into equation **(4)** gives:

$\frac{51u}{4} - 3y = 2u \Rightarrow y = \frac{43u}{12}$

7 a i Use $v^2 = u^2 + 2as$ downwards with $u = 0, s = 0.4$ and $a = g = 9.8$ to find the speed of approach for the first bounce:

$v^2 = 2g \times 0.4 = 7.84$

$v = 2.8\,\text{m s}^{-1}$

Newton's law of restitution gives speed of rebound from floor as $0.7 \times 2.8 = 1.96\,\text{m s}^{-1}$

Use $v^2 = u^2 + 2as$ upwards with $v = 0, u = 1.96$ and $a = -g$ to find the height of the first bounce:

$0 = 1.96^2 - 2gs$

$s = \dfrac{1.96^2}{19.6} = 0.196 = 19.6 \text{ cm}$

ii Use $v^2 = u^2 + 2as$ downwards with $u = 0, s = 0.196$ and $a = g = 9.8$ to find the speed of approach for the second bounce:

$v^2 = 2g \times 0.196 = 3.8416$

$v = 1.96\,\text{m s}^{-1}$ [This result can also be directly deduced]

Following second collision with floor, the ball rebounds with speed $0.7 \times 1.96 = 1.372\,\text{m s}^{-1}$

Use $v^2 = u^2 + 2as$ upwards with $v = 0, u = 1.372$ and $a = -g$

$0 = 1.96^2 - 2gs$

$s = \dfrac{1.372^2}{19.6} = 0.096\,04 = 9.604 \text{ cm}$

b The ball continues to bounce (for an infinite amount of time) with its height decreasing by a common ratio each time.

c Ratio of heights of successive bounces is $\dfrac{9.604}{19.6} = 0.49$

Total distance travelled $= 0.4 + (2 \times 0.196 + 2 \times 0.196 \times 0.49 + 2 \times 0.196 \times 0.49 \times 0.49 \ldots)$

$= 0.4 + \dfrac{2 \times 0.196}{1 - 0.49}$ (using the sum of an infinite geometric series)

$= 1.17\,\text{m}$ (3 s.f.)

d The ball loses energy following every bounce, so an infinite number of bounces would be unrealistic.

8 a Use $v^2 = u^2 + 2as$ downwards with $u = 0, s = H$ and $a = g$

$v^2 = 2gH \Rightarrow v = \sqrt{2gH}$

Newton's law of restitution gives speed of separation from plane as $e\sqrt{2gH}$

Let the height to which the ball rebounds after the first bounce be h_1

Use $v^2 = u^2 + 2as$ upwards with $v = 0, u = e\sqrt{2gH}$, $a = -g$ and $s = h_1$

$0 = 2gHe^2 - 2gh_1$

$\Rightarrow h_1 = e^2 H$

b Let the height to which the ball rebounds after the second bounce be h_2

Before the second bounce, the ball drops from a height h_1

So using the result from part **a**, $h_2 = e^2 h_1$

So $h_2 = e^2 h_1 = e^2(e^2 H) = e^4 H$

c Let the total distance travelled by the ball be d, then

$$
\begin{aligned}
d &= H + 2h_1 + 2h_2 + \ldots \\
&= H + 2e^2 H + 2e^4 H + \ldots \\
&= H + 2e^2 H(1 + e^2 + e^4 + \ldots)
\end{aligned}
$$

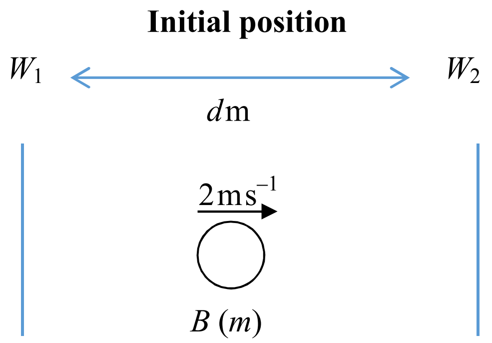
$2e^2 H(1 + e^2 + e^4 + \ldots)$ is an infinite geometric series with first term $a = 2e^2 H$ and common ratio $r = e^2$, so

$$S_\infty = \frac{a}{1-r} = \frac{2e^2 H}{1-e^2}$$

Therefore

$$
\begin{aligned}
d &= H + 2e^2 H(1 + e^2 + e^4 + \ldots) = H + \frac{2e^2 H}{1-e^2} \\
&= \frac{H(1-e^2) + 2e^2 H}{1-e^2} = \frac{H + e^2 H}{1-e^2} \\
&= \frac{H(1+e^2)}{1-e^2}
\end{aligned}
$$

9



From $O \rightarrow W_2$, B travels at a speed of 2m s^{-1} through a distance $\frac{d}{2}$ m.

So the time taken is $\frac{\text{distance}}{\text{speed}} = \frac{\frac{d}{2}}{2} = \frac{d}{4}$

B then rebounds with speed $2e_2\,\text{m s}^{-1}$.

From $W_2 \rightarrow W_1$, B travels at a speed of $2e_2\,\text{m s}^{-1}$ through a distance dm.

So the time taken is $\frac{\text{distance}}{\text{speed}} = \frac{d}{2e_2}$

B then rebounds with speed $2e_2e_1\,\text{m s}^{-1}$.

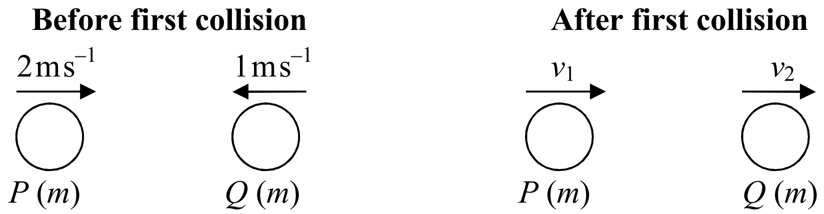
From $W_1 \rightarrow W_2$, B travels at a speed of $2e_2e_1\,\text{m s}^{-1}$ through a distance dm.

So the time taken is $\frac{\text{distance}}{\text{speed}} = \frac{d}{2e_2e_1}$

Therefore, the total time taken is $\frac{d}{4} + \frac{d}{2e_2} + \frac{d}{2e_2e_1} = \frac{d}{2}\left(\frac{1}{2} + \frac{1}{e_1} + \frac{1}{e_1e_2}\right)$ seconds.

Challenge

Consider the initial collision of particles P and Q:



Using conservation of linear momentum for the system $(\rightarrow)$:

$2m - m = mv_1 + mv_2$

$\Rightarrow v_1 + v_2 = 1$ **(1)**

Using Newton's law of restitution:

$e = 0.5 = \dfrac{v_2 - v_1}{2-(-1)} = \dfrac{v_2 - v_1}{3}$

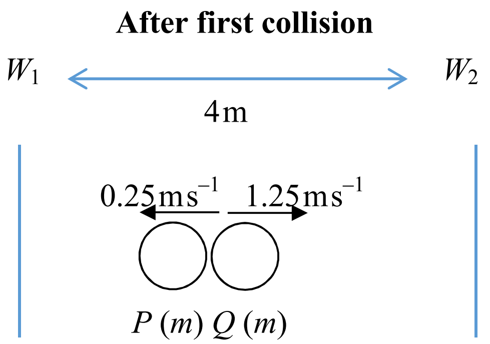
$\Rightarrow v_2 - v_1 = 1.5$ **(2)**

Adding equations **(1)** and **(2)** gives:

$2v_2 = 2.5 \Rightarrow v_2 = 1.25\,\text{m s}^{-1}$

Substituting into equation **(1)** gives:

$1.25 - v_1 = 1.5 \Rightarrow v_1 = -0.25\,\text{m s}^{-1}$



As P travels from O to W_1, the time taken is $\dfrac{\text{distance}}{\text{speed}} = \dfrac{2}{0.25} = 8\,\text{s}$

As Q travels from O to W_2, the time taken is $\dfrac{\text{distance}}{\text{speed}} = \dfrac{2}{1.25} = 1.6\,\text{s}$

Q rebounds with speed $1.25 \times 0.4 = 0.5\,\text{m s}^{-1} 1.25 \times 0.4 = 0.5\ \text{m s}^{-1}$.

Now let t be the time of the second collision, and suppose both particles collide at a distance d from W_2

Challenge continued

Then for particle P: $d = 2 + 0.25t$
And for particle Q: $d = 0.5(t - 1.6)$

So $2 + 0.25t = 0.5(t - 1.6)$

$0.25t = 2 + 0.8 = 2.8$

$t = 11.2$ seconds

But it only takes 8 seconds for P to travel to W_1, so P will hit W_1 before colliding with Q for a second time.

Therefore P hits W_1 before colliding with Q for a second time.