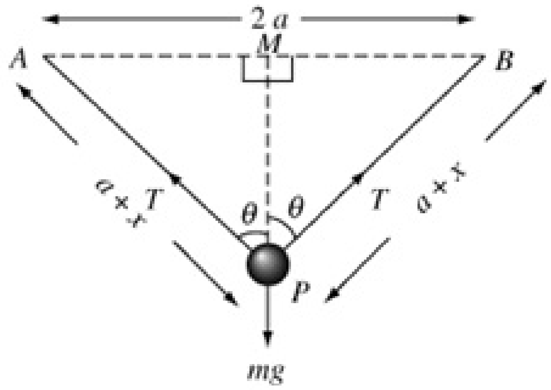
Elastic strings and springs Mixed exercise 3

1



$(\uparrow)\ 2T\cos\theta = mg$ **(1)**

By Hooke's law

$T = \dfrac{15mgx}{16a}$ **(2)**

$\sin\theta = \dfrac{a}{a+x}$ **(3)**

a If $\cos\theta = \dfrac{4}{5}$, $T = \dfrac{5mg}{8}$ from **(1)**

so, $\dfrac{5mg}{8} = \dfrac{15mgx}{16a}$ from **(2)**

$\dfrac{2a}{3} = x$

If $\cos\theta = \dfrac{4}{5}$, then $\sin\theta = \dfrac{3}{5}$

$\sin\theta = \dfrac{a}{a + \frac{2a}{3}}$ from **(3)**

$= \dfrac{3}{5}$

which is true. So, $\cos\theta = \dfrac{4}{5}$.

b Work done on particle $=$ overall gain in energy

$=$ P.E. gain $-$ E.P.E. loss

$PM = (a+x)\cos\theta$

$= \left(a + \dfrac{2a}{3}\right)\dfrac{4}{5}$

$= \dfrac{4a}{3}$

$\therefore$ P.E. gain $= mg\dfrac{4a}{3}$

E.P.E. loss = initial E.P.E. $-$ final E.P.E.

$= \dfrac{15mg}{16\times 2a}\left(2\times\left(\dfrac{2a}{3}\right)^2 - 0^2\right)$

$= \dfrac{15mg\times 2\times 4a^2}{16\times 2a\times 9}$

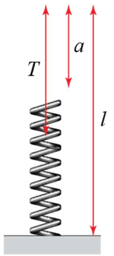
$= \dfrac{5mga}{12}$

So, work done $= \dfrac{4mga}{3} - \dfrac{5mga}{12}$

$= \dfrac{mga}{12}(16-5)$

$= \dfrac{11mga}{12}$

2 Let l be the natural length of the spring.
Let λ be the modulus of the spring.

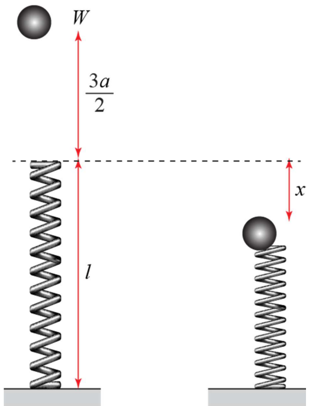


$(\uparrow)\ T = W$

by Hooke's law,

$$T = \frac{\lambda a}{l}$$

$$\therefore\ W = \frac{\lambda a}{l} \quad \text{i.e.} \quad \frac{W}{a} = \frac{\lambda}{l}$$



Using conservation of energy,

P.E. loss of W = E.P.E. gain of spring

$$W\left(\frac{3a}{2} + x\right) = \frac{\lambda x^2}{2l}$$

$$\text{so, } W\left(\frac{3a}{2} + x\right) = \frac{Wx^2}{2a}$$

Substitute for $\frac{\lambda}{l}$ from above.

$$3a^2 + 2ax = x^2$$

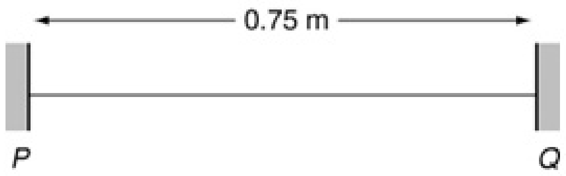
$$0 = x^2 - 2ax - 3a^2$$

$$0 = (x - 3a)(x + a)$$

$$\therefore x = 3a \text{ or } -a$$

$\therefore$ maximum compression is $3a$

3 a

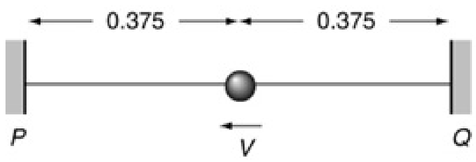


$x = 0.75 - 0.5 = 0.25$

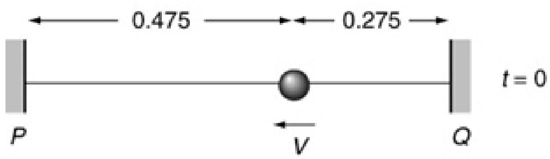
by Hooke's law, $15 = \frac{\lambda \times 0.25}{0.5}$

$\Rightarrow \lambda = 30\,\text{N}$

b



(final)



(initial)

Using conservation of energy

K.E. gain = E.P.E. loss

Initial E.P.E. $= \frac{30}{2 \times 0.25}((0.475 - 0.25)^2 + (0.275 - 0.25)^2)$

E.P.E. loss = initial E.P.E. − final E.P.E.

$$= \frac{30}{2 \times 0.25}(0.225^2 + 0.025^2 - 2 \times 0.125^2)$$

$$= 60(0.05125 - 0.03125)$$

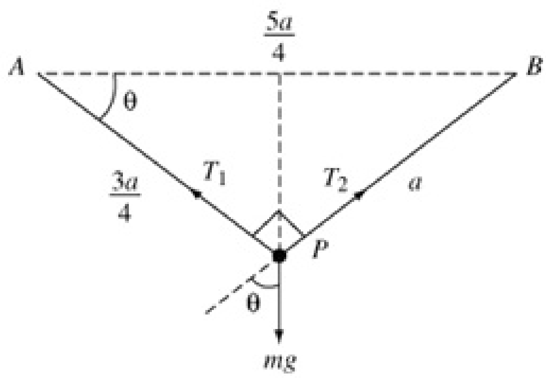
$$= 1.2\,\text{J}$$

$$\frac{1}{2} \times \frac{1}{2} \times v^2 = 1.2$$

So, $v^2 = 4.8$

$v = 2.19\text{ m s}^{-1}$ (3 s.f.)

4 Triangle *ABP* is a 3,4,5 triangle, so angle *APB* is a right angle.



$\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$

$(\uparrow)\ T_1\sin\theta + T_2\cos\theta = mg$

$$\frac{4}{5}T_1 + \frac{3}{5}T_2 = mg$$

$$4T_1 + 3T_2 = 5mg \quad \textbf{(1)}$$

$(\rightarrow)\ T_1\cos\theta = T_2\sin\theta$

$$\frac{3}{5}T_1 = \frac{4}{5}T_2$$

$$T_1 = \frac{4}{3}T_2 \quad \textbf{(2)}$$

Substituting from **(2)** into **(1)**:

$$\frac{16}{3}T_2 + 3T_2 = 5mg$$

$$25T_2 = 15mg$$

$$T_2 = \frac{3mg}{5}$$

From Hooke's law,

$$T_2 = \frac{\lambda x}{l} = \frac{\lambda(a-l)}{l}$$

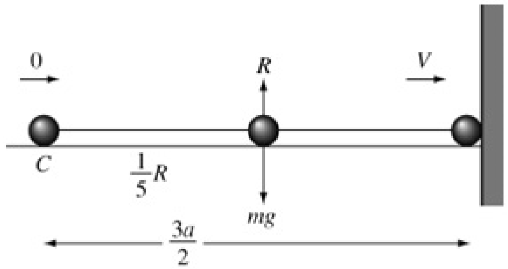
$$\frac{3mg}{5} = \lambda\left(\frac{a}{l} - 1\right)$$

$$\frac{3mg}{5\lambda} + 1 = \frac{a}{l}$$

$$\frac{3mg + 5\lambda}{5\lambda} = \frac{a}{l}$$

$$l = \frac{5\lambda a}{3mg + 5\lambda}$$

5 a



$(\uparrow)\ R = mg$

$\therefore$ Friction $= \frac{1}{5}mg$

Work done against friction = overall loss in energy

= E.P.E. loss − K.E. gain

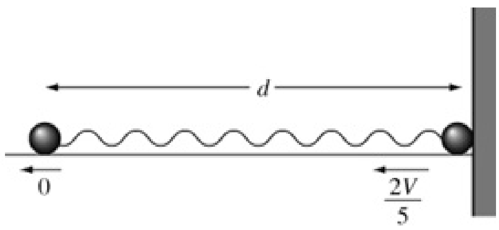
$$\frac{1}{5}mg\frac{3a}{2} = \frac{5mg\left(\frac{a}{2}\right)^2}{2a} - \frac{1}{2}mV^2$$

$$\frac{3ag}{5} = \frac{5ag}{4} - V^2$$

$$V^2 = \frac{5ag}{4} - \frac{3ag}{5} = \frac{ag(25-12)}{20}$$

$$V = \sqrt{\frac{13ag}{20}}$$

b



Friction will be the same.
Assume string is still slack when ball comes to rest.

Work done against friction = K.E. loss

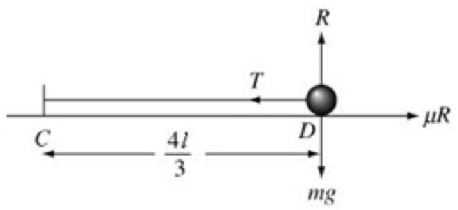
$$\frac{1}{5}mg\,d = \frac{1}{2}m\left(\frac{2V}{5}\right)^2 = \frac{1}{2}m\frac{4V^2}{25}$$

$$\frac{1}{5}gd = \frac{1}{2}\times\frac{4}{25}\times\frac{13ag}{20}$$

$$d = \frac{13a}{50}$$

As d is less than a, the assumption that the string is still slack is valid.

6 a



$(\uparrow) R = mg \quad (\rightarrow) \mu R = T$

$\mu mg = T$

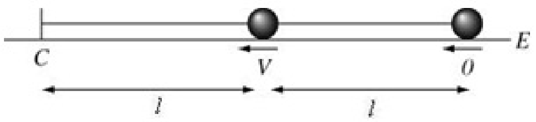
by Hooke's law,

$T = \frac{2mg}{l} \times \frac{l}{3} = \frac{2mg}{3}$

$\therefore \quad \mu mg = \frac{2mg}{3}$

$\mu = \frac{2}{3}$

b



Work done against friction = overall loss in energy

= E.P.E. loss – K.E. gain

$$\frac{2}{3} mg\, l = \frac{2mgl^2}{2l} - \frac{1}{2} mV^2$$

$$\frac{1}{2} V^2 = gl - \frac{2}{3} gl = \frac{1}{3} gl$$

$$V^2 = \frac{2}{3} gl$$

$$V = \sqrt{\frac{2gl}{3}}$$

c String is now slack.

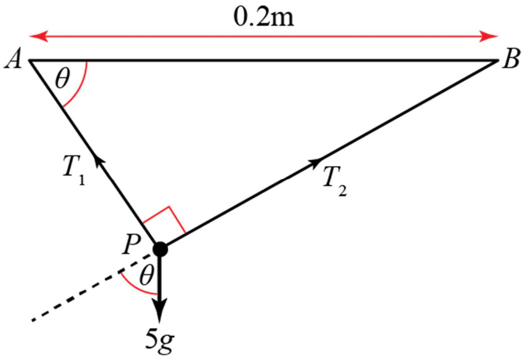
Work done against friction = K.E. loss

$$\frac{2}{3} mg\, d = \frac{1}{2} m \times \frac{2}{3} gl$$

$$d = \frac{1}{2} l$$

Total distance travelled is $\frac{3l}{2}$.

7 a



extension of AP $(x_1) = 0.2\cos\theta - 0.15$

extension of BP $(x_2) = 0.2\sin\theta - 0.05$

$$\therefore \text{ratio is} \quad \frac{x_1}{x_2} = \frac{0.2\cos\theta - 0.15}{0.2\sin\theta - 0.05} \times \frac{20}{20}$$

$$= \frac{4\cos\theta - 3}{4\sin\theta - 1}$$

b $(\nearrow)$ along $PB: T_2 = 5g\cos\theta$

$(\swarrow)$ along $PA: T_1 = 5g\sin\theta$

$$\text{so,} \frac{T_2}{T_1} = \frac{\cos\theta}{\sin\theta}$$

$$\frac{\lambda x_2}{0.05} \times \frac{0.15}{\lambda x_1} = \frac{\cos\theta}{\sin\theta}$$

$$\frac{3x_2}{x_1} = \frac{\cos\theta}{\sin\theta}$$

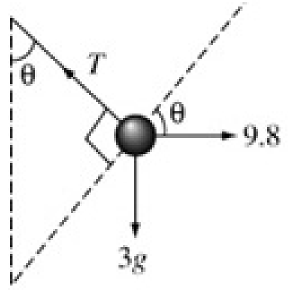
$$\text{i.e.} \ \frac{x_1}{x_2} = \frac{3\sin\theta}{\cos\theta}$$

Using the answer to part **a**:

$$\frac{4\cos\theta - 3}{4\sin\theta - 1} = \frac{3\sin\theta}{\cos\theta}$$

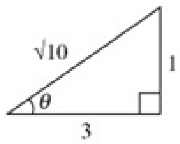
$$3\sin\theta(4\sin\theta - 1) = \cos\theta(4\cos\theta - 3)$$

8 a



(↗ perpendicular to string)

$$9.8\cos\theta = 3g\sin\theta$$



$$\frac{1}{3} = \tan\theta$$

$$\theta = \tan^{-1}\left(\frac{1}{3}\right) = 18.4°$$

b $(\rightarrow)\ T\sin\theta = 9.8$

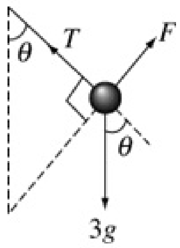
$$T = 9.8\sqrt{10}$$

$$\frac{14.7 \times x}{1} = 9.8\sqrt{10}$$

$$x = \frac{2\sqrt{10}}{3} \approx 2.108\ldots$$

The extension is 2.1 m (2 s.f.).

c



Least force will be perpendicular to the string

$$(\nearrow)\ F = 3g\sin\theta$$

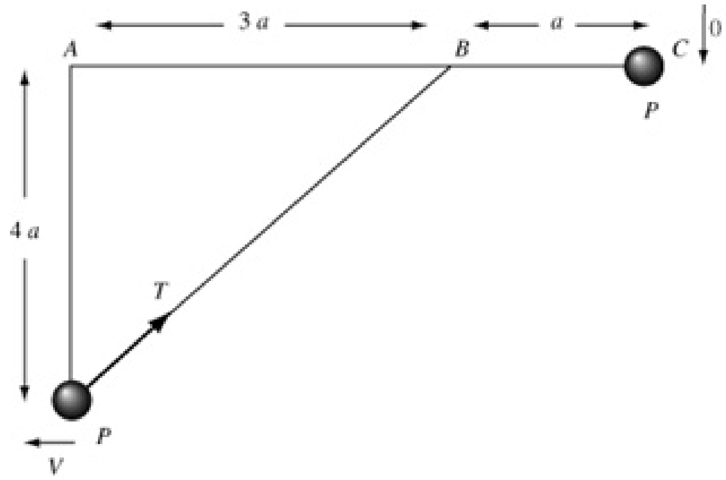
$$= \frac{3g}{\sqrt{10}}$$

$$= \frac{3g\sqrt{10}}{10}$$

$$= 9.297\ldots$$

The least force is 9.3 N (2 s.f.).

9



a By conservation of energy,

$$\text{K.E. gain} + \text{E.P.E. gain} = \text{P.E. loss}$$

$$\frac{1}{2}mV^2 + \left(\frac{mg}{4} \times \frac{x^2}{2a}\right) = mg \times 4a$$

$BP = 5a$ (3,4,5 triangle)

So, $x = 4a$

$$\therefore \quad \frac{1}{2}mV^2 + \left(\frac{mg}{4} \times \frac{16a^2}{2a}\right) = 4mga$$

$$V^2 + 4ga = 8ga$$

$$V^2 = 4ga$$

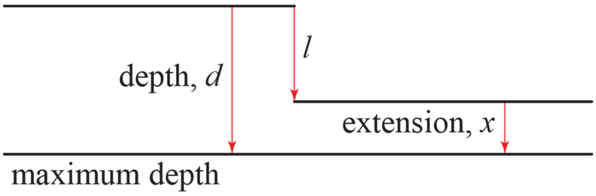
$$V = 2\sqrt{ga}$$

b $x = 4a : T = \frac{mg}{4} \times \frac{4a}{a}$

$$= mg$$

Challenge

We define the variables for the problem as in the following diagram;



a Now, applying conservation of elastic potential energy and gravitational potential energy, we find that:

$$mgd = \frac{1}{2} \cdot \frac{\lambda}{l} \cdot (d - l)^2$$
$$\Rightarrow mgld = \frac{\lambda}{2}\left(d^2 + l^2 - 2dl\right)$$
$$\Rightarrow \frac{2mgl}{\lambda} d = d^2 + l^2 - 2dl$$

Substituting $k = \frac{mgl}{\lambda}$ and rearranging, we see that:

$$d^2 - 2(l + k)d + l^2 = 0$$
$$\Rightarrow d = \frac{1}{2}\left(2(l + k) \pm 2\sqrt{(l + k)^2 - l^2}\right)$$
$$\Rightarrow d = (l + k) \pm \sqrt{k^2 + 2lk}$$

But we know that *d* must be larger than *l*, else the string wouldn't be taut when the maximum depth was reached, so we should take the positive square root, giving the result.

b **i** Suppose the jumper had an initial downwards velocity, *v*.

Then they would have an initial kinetic energy $\frac{1}{2}mv^2$ in the downwards direction, in addition to the initial GPE of mgd. So the distance the jumper falls increases.

ii If we included air resistance, the frictional force would do work on the jumper as they fell. Then the energy balance is GPE + EPE + Work done by friction = 0. This results in a reduced GPE, decreasing the maximum distance the jumper falls.