The simplex algorithm Mixed exercise

1 **a** There are no negative numbers in the profit now.

b $P+\frac{3}{2}x+\frac{3}{4}r=840$

So $P=840-\frac{3}{2}x-\frac{3}{4}r$

Increasing x or r would decrease P.

c **i** Maximum profit $=$ £840

ii Optimum number of $A=0,\ B=56$ and $C=75$

2 **a** Maximise $P=14x+20y+30z$

Subject to:

$5x+8y+10z+r=25\,000$

$5x+6y+15z+s=36\,000$

where r and s are slack variables $x,y,z,r,s\geqslant 0$

b

b.v.	x	y	z	r	s	**value**
r	$1\frac{2}{3}$	4	0	1	$-\frac{2}{3}$	1000
z	$\frac{1}{3}$	$\frac{2}{5}$	1	0	$\frac{1}{15}$	2400
P	-4	-8	0	0	2	72 000

b.v.	x	y	z	r	s	**value**	**Row operations**
y	$\frac{5}{12}$	1	0	$\frac{1}{4}$	$-\frac{1}{6}$	250	R1 $\div$ 4
z	$\frac{1}{6}$	0	1	$-\frac{1}{10}$	$\frac{2}{15}$	2300	R2 $-\frac{2}{5}$R1
P	$-\frac{2}{3}$	0	0	2	$\frac{2}{3}$	74 000	R3 $+$ 8R1

b.v.	x	y	z	r	s	**value**	**Row operations**
x	1	$2\frac{2}{5}$	0	$\frac{3}{5}$	$-\frac{2}{5}$	600	R1 $\div\frac{5}{12}$
z	0	$-\frac{2}{5}$	1	$-\frac{1}{5}$	$\frac{1}{5}$	2200	R2 $-\frac{1}{6}$R1
P	0	$1\frac{3}{5}$	0	$2\frac{2}{5}$	$\frac{2}{5}$	74 400	R3 $+\frac{2}{3}$R1

i $x=600\quad y=0\quad z=2200$

c **ii** Profit is $=$ £744

iii The solution is optimal since there are no negative numbers in the profit row.

3 **a** $\frac{1}{5}(x+y+z) \geqslant y \Rightarrow -x+4y-z \leqslant 0$

$60x+100y+160z \leqslant 2000 \Rightarrow 3x+5y+8z \leqslant 100$

$x, y, z \geqslant 0$

b $S = 2x+4y+6z$

c There are three variables.

d

b.v.	x	y	z	r	t	value
r	-1	4	-1	1	0	0
t	3	5	8	0	1	100
S	-2	-4	-6	0	0	0

e

b.v.	x	y	z	r	t	value	Row operations
r	$-\frac{5}{8}$	$4\frac{5}{8}$	0	1	$\frac{1}{8}$	$12\frac{1}{2}$	$\text{R}1+\text{R}2$
z	$\frac{3}{8}$	$\frac{5}{8}$	1	0	$\frac{1}{8}$	$12\frac{1}{2}$	$\text{R}2 \div 8$
S	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{3}{4}$	75	$\text{R}3+6\text{R}2$

f

b.v.	x	y	z	r	t	value	Row operations
y	$-\frac{5}{37}$	1	0	$\frac{8}{37}$	$\frac{1}{37}$	$2\frac{26}{37}$	$\text{R}1 \div 4\frac{5}{8}$
z	$\frac{17}{37}$	0	1	$-\frac{5}{37}$	$\frac{4}{37}$	$10\frac{30}{37}$	$\text{R}2-\frac{5}{8}\text{R}1$
S	$\frac{8}{37}$	0	0	$\frac{2}{37}$	$\frac{28}{37}$	$75\frac{25}{37}$	$\text{R}3+\frac{1}{4}\text{R}1$

g There are no negative numbers in the objective row.

h 0 small, 2 medium and 11 large tables (seating 74) at a cost of £1960.

4 **a** *Material*

$(\times 100)\ 0.05x+0.08y \leqslant 20$

$5x+8y \leqslant 2000$

Time

$(\div 4)\ 12x+8y \leqslant 2880$

$3x+2y \leqslant 720$

where $x \geqslant 0, y \geqslant 0$

4 b

b.v.	*x*	*y*	*r*	*s*	**value**
r	5	8	1	0	2000
s	3	$\textcircled{2}$	0	1	720
P	−1.5	−1.75	0	0	0

c

b.v.	*x*	*y*	*r*	*s*	**value**	**Row operations**
Y	$\frac{5}{8}$	1	$\frac{1}{8}$	0	250	$R1 \div 8$
s	$\textcircled{1\frac{3}{4}}$	0	$-\frac{1}{4}$	1	220	R2
P	$-\frac{13}{32}$	0	$\frac{7}{32}$	0	$437\frac{1}{2}$	$R3 - 1\frac{3}{4}R1$

b.v.	*x*	*y*	*r*	*s*	**value**	**Row operations**
y	0	1	$\frac{3}{14}$	$-\frac{5}{14}$	$171\frac{3}{7}$	$R1 - \frac{5}{8}R2$
x	1	0	$-\frac{1}{7}$	$\frac{4}{7}$	$125\frac{5}{7}$	$R2 \div 1\frac{3}{4}$
P	0	0	$\frac{9}{56}$	$\frac{13}{56}$	$488\frac{4}{7}$	$R3 \div 1\frac{13}{32}R2$

Optimal solution $x = 125\frac{5}{7}$ $y = 171\frac{3}{7}$

Integer solution needed, so point testing gives $x = 126$ $y = 171$with a total profit of £488.25

d The first point is A if *y* is increased first (*D* if *x* increased first).
The second point is C.

5 a Watchmaker

$$54x + 72y + 36z \leqslant 1800$$

(÷18)

$$3x + 4y + 2z \leqslant 100$$

Fitter

$$60x + 36y + 48z \leqslant 1500$$

(÷12)

$$5x + 3y + 4z \leqslant 125$$

b $P = 12x + 24y + 20z$

c

b.v.	*x*	*y*	*z*	*r*	*s*	**value**
r	3	4	2	1	0	100
s	5	$\textcircled{3}$	4	0	1	125
P	−12	−24	−20	0	0	0

5 d

b.v.	x	y	z	r	s	**value**	**Row operations**
y	$\frac{3}{4}$	1	$\textcircled{\frac{1}{2}}$	$\frac{1}{4}$	0	25	$R1 \div 4$
s	$2\frac{3}{4}$	0	$2\frac{1}{2}$	$-\frac{3}{4}$	1	50	$R2-3R1$
P	6	0	-8	6	0	600	$R3+24R1$

e

b.v.	x	y	z	r	s	**value**	**Row operations**
y	$\frac{1}{5}$	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	15	$R1-\frac{1}{2}R2$
z	$\frac{11}{10}$	0	1	$-\frac{3}{10}$	$\frac{2}{5}$	20	$R2 \div 2\frac{1}{2}$
P	$14\frac{4}{5}$	0	0	$3\frac{3}{5}$	$3\frac{1}{5}$	760	$R3+8R2$

There are no negative numbers in the profit row.

f Type A = 0 Type B = 15 Type C = 20

Profit = £760

6 a Maximise $P=14x+12y+13z$

Subject to:

Carving $2x+2.5y+1.5z \leqslant 8 \Rightarrow 4x+5y+3z \leqslant 16$

Sanding $25x+20y+30z \leqslant 120 \Rightarrow 5x+4y+6z \leqslant 24$

$x, y, z \geqslant 0$

b r and s are numbers which indicate the slack time.

Profit: $P-14x-12y-13z=0$

Constraints:

$4x+5y+3z+r=16$

$5x+4y+6z+s=24$

b.v.	x	y	z	r	s	**value**
r	4	5	3	1	0	16
s	$\textcircled{5}$	4	6	0	1	24
P	-14	-12	-13	0	0	0

c

b.v.	x	y	z	r	s	**value**	**Row operations**
x	1	$\frac{5}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	0	4	$R1 \div 4$
s	0	$-\frac{9}{4}$	$\frac{9}{4}$	$-\frac{5}{4}$	1	4	$R2-5R1$
P	0	$\frac{11}{2}$	$-\frac{5}{2}$	$\frac{7}{2}$	0	56	$R3+14R1$

6 d From a zero stock situation, if we increase the number of lions to 4, we are increasing the profit from 0 to £56.

7 a $3x+2y+s_1=15$

$2x+5y+s_2=20$

$y-s_3+a_1=2$

b Want to maximise $I=-a_1=y-s_3-2$

c

Basic variable	x	y	s_1	s_2	s_3	a_1	Value
s_1	3	2	1	0	0	0	15
s_2	2	5	0	1	0	0	20
a_1	0	1	0	0	–1	1	2
P	–1	–3	0	0	0	0	0
I	0	–1	0	0	1	0	–2

d

Basic variable	x	y	s_1	s_2	s_3	a_1	Value	θ values
s_1	3	2	1	0	0	0	15	$\frac{15}{2}$
s_2	2	5	0	1	0	0	20	4
a_1	0	1	0	0	–1	1	2	1
P	–1	–3	0	0	0	0	0	
I	0	–1	0	0	1	0	–2	

Basic variable	x	y	s_1	s_2	s_3	a_1	Value	Row operations
s_1	3	0	1	0	2	–2	11	R1 – 2R3
s_2	2	0	0	1	5	–5	10	R2 – 5R3
y	0	1	0	0	–1	1	2	R3
P	–1	0	0	0	–3	3	6	R4 + 3R3
I	0	0	0	0	0	1	0	R5 + R3

$I=0$ so we have a basic feasible solution: $P=6$ when $x=0, y=2, s_1=11, s_2=10$

7 e Truncate final tableau from part **d** to set up initial tableau for the second stage.

Basic variable	x	y	s_1	s_2	s_3	Value
s_1	3	0	1	0	2	11
s_2	2	0	0	1	5	10
y	0	1	0	0	–1	2
P	–1	0	0	0	–3	6

8 a The only negative value in profit row is in column x. θ values are $\frac{150}{2}=75, \frac{180}{1}=180, \frac{70}{1}=0$

The smallest is in row a_1, so 1 in column x, row a_1 is the pivot.

b

Basic variable	x	y	z	s_1	s_2	s_3	a_1	Value	Row operations
s_1	0	0	1	1	2	–2	–2	10	R1 – 2R3
s_2	0	1	–1	0	1	–1	–1	110	R2 – R3
x	1	0	0	0	0	1	1	70	R3
P	0	0	1	0	M	$M+6$	$2M+1$	300	R4 + R3

Solution is optimal since all values in the profit row are non-negative.
$P=300$ when $x=70, y=0, z=0, s_1=10, s_2=110, s_3=0$

9 a $a_1 \neq 0$, so the solution is not feasible

b The most negative entry in the profit row is in column x. θ values are $4, \frac{5}{2}, 4,$ of which $\frac{5}{2}$ is the least positive theta value. Hence the pivot is 2 in row s_2, column x.

c

	x	y	z	s_1	s_2	s_3	a_1	Value	Row operation
s_1	0	$-\frac{1}{2}$	$\frac{7}{2}$	1	$-\frac{1}{2}$	0	0	$\frac{3}{2}$	R1 – R2
y	1	$\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	$\frac{5}{2}$	R2 ÷ 2
a_1	0	−4	−1	0	−1	−1	1	3	R3 − 2R2
P	0	$\frac{5}{2}+4\text{M}$	$\enclose{circle}{-\frac{7}{2}}+\text{M}$	0	$\frac{1}{2}+\text{M}$	M	0	$\frac{5}{2}-3\text{M}$	R4 + (1 + 2M)R2

10 a It contains a "$\geqslant$" constraint, i.e. the origin is not a feasible solution.

b Pie A gives most profit, so optimal solution should have as big x as possible. However, due to a constraint, we need $z \geqslant 200$ Hence, optimal solution is $x=600, y=0, z=200$

10 c M represents an arbitrarily large positive number.

d Rewrite the constraints using slack, surplus and artificial variables.

$$x + y + z + s_1 = 800$$
$$z - s_2 + a_1 = 200$$
$$2x + 2y + z + s_3 = 1200$$
$$4y + 5z - s_4 + a_2 = 1000$$

Write down the optimal function suitable for Big-M method.

$$P = 100x + 80y + 60z - M(a_1 + a_2) = 100x + 80y + 60z - M(200 - z + s_2 + 1000 - 4y - 5z + s_4)$$
$$\Rightarrow P - 100x - (80 + 4M)y - (60 + 6M)z + Ms_2 + Ms_4 = -1200M$$

Basic variable	x	y	z	s_1	s_2	s_3	s_4	a_1	a_2	Value
s_1	1	1	1	1	0	0	0	0	0	800
a_1	0	0	1	0	-1	0	0	1	0	200
s_3	2	2	1	0	0	1	0	0	0	1200
a_2	0	4	5	0	0	0	-1	0	1	1000
P	-100	$-(80+4M)$	$-(60+6M)$	0	M	0	M	0	0	$-1200M$

Formulate initial tableaux.

e The most negative value in the objective row is $-(60 + 60M)$ in the z column, so this is the pivot column. θ values are $800, 200, 1200, \frac{1000}{5} = 200$, so either 1 in row a_1 or 5 in row a_2 can form a pivot.

Divide pivot row by the pivot values. Then, using row operations remove the z entries from all rows except the pivot row so that the pivot column only contains 1s and 0s. Continue this way (choose next pivot and apply row operations). When there are no negative entries in the objective row we terminate as the solution is optimal.

Challenge

P = 28.5 when $x = \frac{9}{2}, y = \frac{1}{4}, z = 7$