The simplex algorithm 7D

1 a Rewrite constraints using slack, surplus and artificial variables.

$x+3y+s_1=15$

$x-y-s_2+a_1=11$

$x, y, s_1, s_2, a_1 \geq 0$

Let $I=-a_1=-11+x-y-s_2$

Formulate initial tableau and iterate.

Basic variable	x	y	s_1	s_2	a_1	Value	θ values
s_1	1	3	1	0	0	15	15
a_1	1	−1	0	−1	1	11	11
P	−2	−1	0	0	0	0	
I	−1	1	0	1	0	−11	

Basic variable	x	y	s_1	s_2	a_1	Value	Row operations
s_1	0	4	1	1	4	1	R1 – R2
x	1	−1	0	−1	1	11	old R2
P	0	−3	0	−2	2	22	R3 + 2R2
I	0	0	0	0	1	0	R4 + R2

$I=0$ so basic feasible solution has been found. Formulate initial tableau and solve the problem using simplex algorithm.

Basic variable	x	y	s_1	s_2	Value	θ values
s_1	0	4	1	1	4	1
x	1	−1	0	−1	11	−11
P	0	−3	0	−2	22	

Basic variable	x	y	s_1	s_2	Value	Row operations
y	0	1	$\frac{1}{4}$	$\frac{1}{4}$	1	Old R1 ÷ 4
x	1	0	$\frac{1}{4}$	$-\frac{3}{4}$	12	R2 + R1
P	0	0	$\frac{3}{4}$	$-\frac{5}{4}$	25	R3 + 3R1

1 a continued

Basic variable	x	y	s_1	s_2	**Value**	θ **values**
y	0	1	$\frac{1}{4}$	$\frac{1}{4}$	1	4
x	1	0	$\frac{1}{4}$	$-\frac{3}{4}$	12	−16
P	0	0	$\frac{3}{4}$	$-\frac{5}{4}$	25	

Basic variable	x	y	s_1	s_2	**Value**	**Row operations**
s_2	0	4	1	1	4	Old R1 × 4
x	1	3	1	0	15	R2 - $\frac{3}{4}$ R1
P	0	5	2	0	30	R3 + $\frac{5}{4}$ R1

Optimal solution is $P = 30$ when $x = 15$, $y = 0$

1 b Rewrite constraints using slack, surplus and artificial variables.

Maximise $P = -C = -x + 3y - z$, subject to

$x + 2y + s_1 = 12$

$2x - y - z - s_2 + a_1 = 10$

$x + y + z - s_3 + a_2 = 6$

$x, y, z, s_1, s_2, s_3, a_1, a_2 \geqslant 0$

Let $I = -(a_1 + a_2) = -16 + 3x$. Formulate initial tableau and iterate.

Basic variable	x	y	z	s_1	s_2	s_3	a_1	a_2	Value	θ values
s_1	1	2	0	1	0	0	0	0	12	12
a_1	2	–1	–1	0	–1	0	1	0	10	5
a_2	1	1	1	0	0	–1	0	1	6	6
P	1	–3	1	0	0	0	0	0	0	
I	–3	0	0	0	1	1	0	0	–16	

Basic variable	x	y	z	s_1	s_2	s_3	a_1	a_2	Value	Row operations
s_1	0	$\frac{5}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	7	R1 – R2
x	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	5	R1 ÷ 2
a_2	0	$\frac{3}{2}$	$\frac{3}{2}$	0	$\frac{1}{2}$	–1	$-\frac{1}{2}$	1	1	R3 – R2
P	0	$-\frac{5}{2}$	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	–5	R4 – R2
I	0	$-\frac{3}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$	0	–1	R5 + 3R2

Basic variable	x	y	z	s_1	s_2	s_3	a_1	a_2	Value	θ values
s_1	0	$\frac{5}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	7	$\frac{24}{5}$
a_1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	5	–10
a_2	0	$\frac{3}{2}$	$\frac{3}{2}$	0	$\frac{1}{2}$	–1	$-\frac{1}{2}$	1	1	$\frac{2}{3}$
P	0	$-\frac{5}{2}$	$\frac{3}{2}$	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	–5	
I	0	$-\frac{3}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$	0	–1	

1 b continued

Basic variable	x	y	z	s_1	s_2	s_3	a_1	a_2	Value	Row operations
s_1	0	0	–2	1	$-\frac{1}{3}$	$\frac{5}{3}$	$\frac{1}{3}$	$-\frac{5}{3}$	$\frac{16}{3}$	R1 $-\frac{5}{2}$ R3
x	1	0	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{16}{3}$	R2 $+\frac{1}{2}$ R3
y	0	1	1	0	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$ R3
P	0	0	4	0	$\frac{4}{3}$	$-\frac{5}{3}$	$-\frac{4}{3}$	$\frac{5}{6}$	$-\frac{10}{3}$	R4 $+\frac{5}{2}$ R3
I	0	0	0	0	0	0	1	1	0	R5 $+\frac{3}{2}$ R3

$I = 0$, so we have found a basic feasible solution.

Basic variable	x	y	z	s_1	s_2	s_3	Value	θ values
s_1	0	0	–2	1	$-\frac{1}{3}$	$\frac{5}{3}$	$\frac{16}{3}$	$\frac{16}{5}$
x	1	0	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{16}{3}$	–16
y	0	1	1	0	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$	–1
P	0	0	4	0	$\frac{4}{3}$	$-\frac{5}{3}$	$-\frac{10}{3}$	

Basic variable	x	y	z	s_1	s_2	s_3	Value	Row operations
s_3	0	0	$-\frac{6}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	1	$\frac{16}{5}$	$\frac{3}{5}$ R1
x	1	0	$-\frac{2}{5}$	$\frac{1}{5}$	$-\frac{2}{5}$	0	$\frac{32}{5}$	R2 $+\frac{1}{3}$ R1
y	0	1	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	0	$\frac{14}{5}$	R3 $+\frac{2}{3}$ R1
P	0	0	2	1	1	0	2	R3 $+\frac{5}{3}$ R1

Optimal solution is $C = -2$ ($P = 2$) when $x = \frac{32}{5}, y = \frac{14}{5}, z = 0$

1 c Rewrite constraints as equalities.

$5x + z + s_1 = 16$

$3x + y + z + s_2 = 12$

$x - y + 4z - s_3 + a_1 = 9$

$x, y, z, s_1, s_2, s_3, a_1 \geqslant 0$

Let $I = -a_1 = -9 + x - y + 4z - s_3$ Construct initial tableau and apply two-stage simplex algorithm.

Basic variable	x	y	z	s_1	s_2	s_3	a_1	Value	θ values
s_1	5	0	1	1	0	0	0	16	16
s_2	3	1	1	0	1	0	0	12	12
a_1	1	–1	4	0	0	–1	1	9	$\frac{9}{4}$
P	–3	1	–2	0	0	0	0	0	
I	–1	1	–4	0	0	1	0	–9	

Basic variable	x	y	z	s_1	s_2	s_3	a_1	Value	Row operations
s_1	$\frac{19}{4}$	$\frac{1}{4}$	0	1	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{55}{4}$	R1 – R3
s_2	$\frac{11}{4}$	$\frac{5}{4}$	0	0	1	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{39}{4}$	R2 – R3
z	$\frac{1}{4}$	$-\frac{1}{4}$	1	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{9}{4}$	R3 ÷ 4
P	$-\frac{5}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{9}{2}$	R4 + 2R3
I	0	0	0	0	0	0	1	0	R5 + 4R3

$I = 0$, so we have found a basic feasible solution. Proceed to the second stage.

Basic variable	x	y	z	s_1	s_2	s_3	Value	θ values
s_1	$\frac{19}{4}$	$\frac{1}{4}$	0	1	0	$\frac{1}{4}$	$\frac{55}{4}$	$\frac{55}{19}$
s_2	$\frac{11}{4}$	$\frac{5}{4}$	0	0	1	$\frac{1}{4}$	$\frac{39}{4}$	$\frac{39}{11}$
z	$\frac{1}{4}$	$-\frac{1}{4}$	1	0	0	$-\frac{1}{4}$	$\frac{9}{4}$	9
P	$-\frac{5}{2}$	$\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$\frac{9}{2}$	

1 c continued

Basic variable	x	y	z	s_1	s_2	s_3	Value	Row operations
x	1	$\frac{1}{19}$	0	$\frac{4}{19}$	0	$\frac{1}{19}$	$\frac{55}{19}$	$\frac{4}{19}$R1
s_2	0	$\frac{21}{19}$	0	$-\frac{11}{19}$	1	$\frac{2}{19}$	$\frac{34}{19}$	R2$-\frac{11}{4}$R1
z	0	$-\frac{5}{19}$	1	$-\frac{1}{19}$	0	$-\frac{5}{19}$	$\frac{29}{19}$	R3$-\frac{1}{4}$R1
P	0	$\frac{12}{19}$	0	$\frac{10}{19}$	0	$-\frac{7}{19}$	$\frac{223}{19}$	R4$+\frac{5}{2}$R1

Basic variable	x	y	z	s_1	s_2	s_3	Value	θ values
x	1	$\frac{1}{19}$	0	$\frac{4}{19}$	0	$\frac{1}{19}$	$\frac{55}{19}$	55
s_2	0	$\frac{21}{19}$	0	$-\frac{11}{19}$	1	$\frac{2}{19}$	$\frac{34}{19}$	17
z	0	$-\frac{5}{19}$	1	$-\frac{1}{19}$	0	$-\frac{5}{19}$	$\frac{29}{19}$	$-\frac{29}{5}$
P	0	$\frac{12}{19}$	0	$\frac{10}{19}$	0	$-\frac{7}{19}$	$\frac{223}{19}$	

Basic variable	x	y	z	s_1	s_2	s_3	Value	Row operations
x	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	2	R1$-\frac{1}{19}$ R2
s_2	0	$\frac{21}{2}$	0	$-\frac{11}{2}$	$\frac{19}{2}$	1	17	$\frac{19}{2}$ R2
z	0	$\frac{5}{2}$	1	$-\frac{3}{2}$	$\frac{5}{2}$	0	6	R3$+\frac{5}{19}$ R2
P	0	$\frac{9}{2}$	0	$-\frac{3}{2}$	$\frac{7}{2}$	0	18	R3$+\frac{7}{19}$ R2

Basic variable	x	y	z	s_1	s_2	s_3	Value	θ values
x	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	2	4
s_3	0	$\frac{21}{2}$	0	$-\frac{11}{2}$	$\frac{19}{2}$	1	17	$-\frac{34}{11}$
z	0	$\frac{5}{2}$	1	$-\frac{3}{2}$	$\frac{5}{2}$	0	6	–4
P	0	$\frac{9}{2}$	0	$-\frac{3}{2}$	$\frac{7}{2}$	0	18	

1 c continued

Basic variable	x	y	z	s_1	s_2	s_3	Value	Row operations
s_1	2	–1	0	1	–1	0	4	2R1
s_3	11	5	0	0	4	1	39	$R2+\frac{11}{2}R1$
z	3	1	1	0	1	0	12	$R3+\frac{3}{2}R1$
P	3	3	0	0	2	0	24	$R4+\frac{3}{2}R1$

Optimal $P = 24$ when $x = 0, y = 0, z = 12$

2 Attempt using two-stage simplex method.

Rewrite constraints as equalities.

$5x_1 + 6x_2 + s_1 = 75$

$3x_1 + 4x_2 - s_2 + a_1 = 52$

Let $I = -a_1 = -75 + 5x_1 + 6x_2 - s_2$. Formulate initial tableau and iterate.

Basic variable	x_1	x_2	s_1	s_2	a_1	Value	θ values
s_1	5	6	1	0	0	75	$\frac{25}{2}$
a_1	3	4	0	–1	1	52	13
I	–3	–4	0	1	0	–52	

Basic variable	x_1	x_2	s_1	s_2	a_1	Value	Row operations
x_2	$\frac{5}{6}$	1	$\frac{1}{6}$	0	0	$\frac{25}{2}$	R1 ÷ 6
a_1	$-\frac{1}{3}$	0	$-\frac{2}{3}$	–1	1	2	R2 – 4R1
I	$\frac{1}{3}$	0	$\frac{2}{3}$	0	0	–2	R3 + 4R1

No negative values in bottom row mean that the solution is optimal. However as $I = -2 \neq 0$, there are no feasible solutions.

3 a i Surplus variable represents the amount by which a solution exceeds the minimum possible value of a quantity.

ii Artificial variable is introduced to $\geqslant$ constraints alongside surplus variable s to allow $s \geqslant 0$

b There are no negative values in the bottom row of the tableau, so it is optimal, and $I = 0$, so there is a basic feasible solution.

3 c Truncate tableau from the question.

Basic variable	x	y	z	s_1	s_2	s_3	Value	θ values
s_1	0	0	2	1	0	1	2	2
x	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{29}{2}$	$-\frac{29}{3}$
y	0	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	–1	$\frac{11}{2}$	$-\frac{11}{2}$
P	0	0	0	0	–1	–2	13	

Basic variable	x	y	z	s_1	s_2	s_3	Value	Row operations
s_3	0	0	2	1	0	1	2	R1
x	1	0	$\frac{7}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{35}{2}$	$R2+\frac{3}{2}R1$
y	0	1	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$\frac{15}{2}$	R3+ R1
P	0	0	4	2	–1	0	17	R4+ 2R1

Basic variable	x	y	z	s_1	s_2	s_3	Value	θ values
s_3	0	0	2	1	0	1	2	∞
x	1	0	$\frac{7}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{35}{2}$	–35
y	0	1	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$\frac{15}{2}$	15
P	0	0	4	2	–1	0	17	

Basic variable	x	y	z	s_1	s_2	s_3	Value	
s_3	0	0	2	1	0	1	2	R1
x	1	1	5	$\frac{5}{2}$	0	0	25	$R2+\frac{1}{2}R3$
s_2	0	2	3	2	1	0	15	2R3
P	0	0	7	4	0	0	32	R4+ R3

Optimal solution is $P = 32$ when $x = 25, y = 0, z = 0, s_1 = 0, s_2 = 15, s_3 = 2$