The simplex algorithm 7C

1 a Introduce slack variables:

$2x + 4y + r = 60$

$3x + 5y + s = 40$

$x, y, r, s \geqslant 0$

Draw the initial tableau and iterate:

Basic variable	x	y	r	s	Value	θ values
r	2	4	1	0	60	30
s	3	5	0	1	40	$13\frac{1}{3}$
P	–3	–2	0	0	0	

Basic variable	x	y	r	s	Value	Row operations
r	0	$\frac{2}{3}$	1	$-\frac{2}{3}$	$33\frac{1}{3}$	R1 − 2R2
x	1	$\frac{5}{3}$	0	$\frac{1}{3}$	$13\frac{1}{3}$	R2 ÷ 3
P	0	3	0	1	40	R3 + 3R2

Optimal non-integer solution: $P = 40$, when $x = 13\frac{1}{3}, y = 0, r = 33\frac{1}{3}, s = 0$

Test integer points around.

Points	$2x + 3y \leqslant 60$?	$3x + 5y \leqslant 40$?	In feasible region?	Value
(13,0)	$26 \leq 60$	$39 \leq 40$	yes	39
(14,0)	$28 \leq 60$	$42 > 40$	no	

Maximal solution we have found for integer x, y is $P = 39$, when $x = 13, y = 0$

1 b Introduce slack variables:

$x + 2z + r = 10$

$4x + 3y - 4z + s = 8$

$x, y, z, r, s \geq 0$

Draw the initial tableau and iterate:

Basic variable	x	y	z	r	s	Value	θ values
r	1	0	2	1	0	10	∞
s	4	3	−4	0	1	8	$\frac{8}{3}$
P	−10	−12	−8	0	0	0	

Basic variable	x	y	z	r	s	Value	Row operations
r	1	0	2	1	0	10	R1
y	$\frac{4}{3}$	1	$-\frac{4}{3}$	0	$\frac{1}{3}$	$\frac{8}{3}$	R2 ÷ 3
P	6	0	−24	0	4	32	R3 + 12R2

Basic variable	x	y	z	r	s	Value	θ values
r	1	0	2	1	0	10	5
y	$\frac{4}{3}$	1	$-\frac{4}{3}$	0	$\frac{1}{3}$	$\frac{8}{3}$	−2
P	6	0	−24	0	4	32	

Basic variable	x	y	z	r	s	Value	Row operations
z	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	5	R2 ÷ 2
y	2	1	0	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{8}{3}$	R2 $+\frac{4}{3}$R1
P	18	0	0	12	4	152	R3 + 24R1

Optimal non-integer solution: $P = 152$, when $x = 0, y = 9\frac{1}{3}, z = 5, r = 0, s = 0$

Test integer points around.

Points	$x + 2z \leqslant 10$?	$4x + 3y - 4z \leqslant 8$?	In feasible region?	Value
(0,9,5)	$10 \leqslant 10$	$7 \leqslant 8$	yes	148
(0,10,5)	$10 \leqslant 10$	$10 > 8$	no	

Maximal solution we have found for integer x, y, z is $P = 148$, when $x = 0, y = 9, z = 5$

2 a at most 80 iced doughnuts $\Rightarrow 2x+3y+z \leqslant 80$
at most 140 French whirls $\Rightarrow 4x+2y+3z \leqslant 140$
at most 96 treacle tarts $\Rightarrow 3x+4y+2x \leqslant 96$

b non-negativity $\Rightarrow x, y, z \geqslant 0$

c most negative value in the objective row is –6 $\Rightarrow$ pivot column is z
least positive θ value is 46 in row s $\Rightarrow$ pivot value is 2 in row s, column z

d

Basic variable	x	y	z	r	s	t	Value	Row operations
r	$\frac{3}{8}$	0	0	1	$\frac{1}{4}$	$-\frac{7}{8}$	31	R1 $+\frac{1}{2}$R2
z	$\frac{5}{4}$	0	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	46	R2 ÷ 2
y	$\frac{1}{8}$	1	0	0	$-\frac{1}{4}$	$\frac{3}{8}$	1	R3 $-\frac{1}{2}$R2
P	$\frac{21}{2}$	0	0	0	3	$\frac{7}{2}$	756	R4 + 6R2

It is an optimal solution. From the table $P = 756 - \frac{21}{2}x - 3s - \frac{7}{2}t$, so increasing x, s or t would decrease profit.

e For the optimal solution $x = 0, y = 1, z = 46,$ so they should prepare zero Supreme, one Dreamtime and 46 Perfection boxes.

Challenge

Rewrite the equations using slack variables:

$P - 5x - 2y - 4z = 0$

$2x + y + kz + r = 10$

$x + 4y + z + s = 12$

$4x - 2y + 3z + t = 28$

$x, y, z, r, s, t > 0$

Use tableau to represent the problem at hand

Basic variable	*x*	*y*	*z*	*r*	*s*	*t*	Value	θ Value
r	2	1	k	1	0	0	10	5*
s	1	4	1	0	1	0	12	12
t	4	−2	3	0	0	1	28	7
P	−5	−2	−4	0	0	0	0	

The most negative entry in the objective row lies in the x column, so we use this column to calculate the θ values. We find the smallest positive θ in the r row.

Basic variable	*x*	*y*	*z*	*r*	*s*	*t*	Value	Row operations
x	1	$\frac{1}{2}$	$\frac{k}{2}$	$\frac{1}{2}$	0	0	5	R1÷2
s	0	$3\frac{1}{2}$	$1-\frac{k}{2}$	$-\frac{1}{2}$	1	0	7	R2 – R1
t	0	−4	$3-\frac{4k}{2}$	−2	0	1	8	R3 – 4R1
P	0	$\frac{1}{2}$	$-4+\frac{5k}{2}$	$\frac{5}{2}$	0	0	25	R4 + 5R1

We know that the optimal solution is obtained after just one iteration of the simplex algorithm, so it must be true that all values in the objective row are non-negative and we have found the optimal solution. So it must be that:

$-4 + \frac{5k}{2} \geqslant 0$

$5k \geqslant 8$

$k \geqslant \frac{8}{5}$

So for the optimal solution to be found after one iteration of the simplex algorithm, we need $k \geqslant \frac{8}{5}$