Review exercise 1

1 a

a	*b*	*c*	*d*	*e*	*f*	*f* = 0?
645	255	253	2	510	135	No
255	135	1.89	1	135	120	No
135	120	1.13	1	120	15	No
120	15	8	8	120	0	Yes

answer is 15

b The first row would be

255 645 0.40 0 0 255 No

but the second row would then be the same as the first row in the table above. So in effect this new first line would just be an additional row at the start of the solution.

c Finds the Highest Common Factor of *a* and *b*.

2 a Total length $= 10 + 15 + 55 + 40 + 75 + 25 + 55 + 60 + 55 = 390$ cm, so 4 planks are needed

b Bin 1: $10 + 15 + 55$
Bin 2: $40 + 25$
Bin 3: 75
Bin 4: 55
Bin 5: 60
Bin 6: 55
This uses 6 planks.

c By inspection,
Bin 1: $60 + 40 = 100$
Bin 2: $75 + 25 = 100$
Bin 3: $55 + 10 + 15 = 80$
Bin 4: 55
Bin 5: 55
This uses 5 planks.

d There are 5 lengths over 50 cm, so none of these can be paired together. Therefore, minimum of 5 lengths are required.

3 a Bubbling left to right

55 80 25 84 25 34 17 75 3 5 $55 < 80$ so swap
80 55 25 84 25 34 17 75 3 5 $55 > 25$ so leave
80 55 84 25 25 34 17 75 3 5 $25 < 84$ so swap
80 55 84 25 25 34 17 75 3 5 $25 \nless 25$ so leave
80 55 84 25 34 25 17 75 3 5 $25 < 34$ so swap
80 55 84 25 34 25 17 75 3 5 $25 > 17$ so leave
80 55 84 25 34 25 75 17 3 5 $17 < 75$ so swap
80 55 84 25 34 25 75 17 3 5 $17 > 3$ so leave
80 55 84 25 34 25 75 17 5 3 $3 < 5$ so swap

After 1st pass: 80 55 84 25 34 25 75 17 5 3
After 2nd pass: 80 84 55 34 25 75 25 17 5 3
After 3rd pass: 84 80 55 34 75 25 25 17 5 3
After 4th pass: 84 80 55 75 34 25 25 17 5 3
After 5th pass: 84 80 75 55 34 25 25 17 5 3
After 6th pass: 84 80 75 55 34 25 25 17 5 3

No swap in 6th pass, so the list is in order.

b $\frac{55+80+25+84+25+34+17+75+3+5}{100} = 4.03$ so 5 bins are needed.

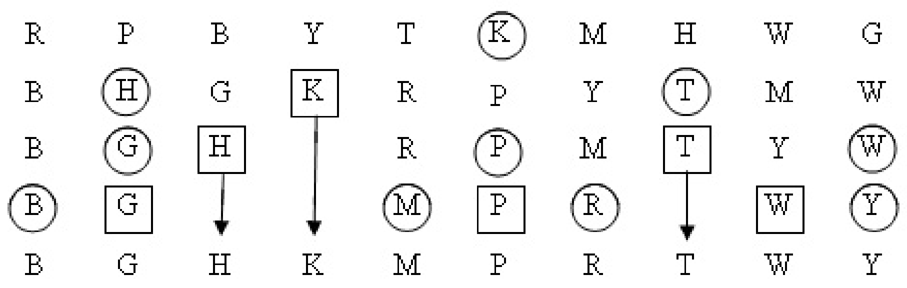
c Using numbers sorted in descending order,
Bin 1: 84 + 5 + 3
Bin 2: 80 + 17
Bin 3: 75 + 25
Bin 4: 55 + 34
Bin 5: 25

4 a After one iteration we have 45 37 18 [46] 56 79 90 81 51

b After one pass we have 56 45 79 46 37 90 81 51 18

c $0.016 \times \left(\frac{3000}{500}\right)^2 = 0.576$ seconds (3 s f)

5 a For example,

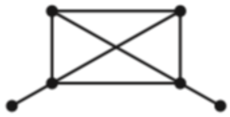


list is in order

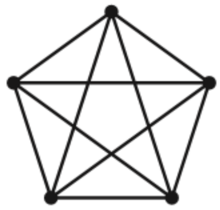
b $0.024 \times \frac{2000 \times \log 2000}{1200 \times \log 1200} = 0.043$ seconds (3 d.p.)

6 a Since the graph is simple, there are no loops, so each of the degree-5 vertices must be joined to each of the other vertices. This means that each of the other vertices has degree at least 2.

b



7 a



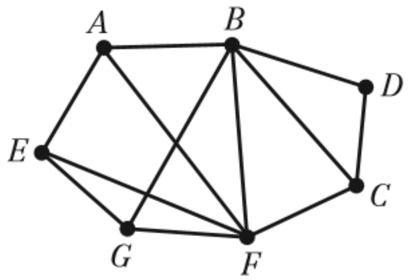
b 4: this includes all the vertices of the original graph and is also a tree.

c 5: cycle that includes every vertex.

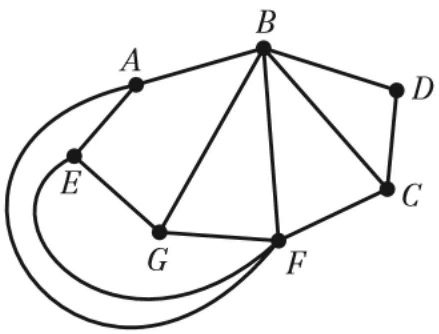
8 a A Hamiltonian cycle is a cycle that includes every vertex.

b *ABDCFGEA*

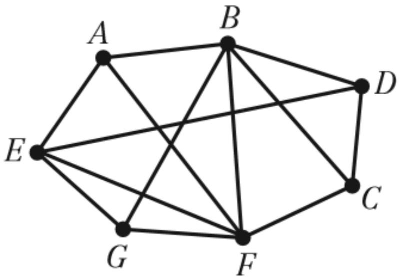
c



There are 5 edges inside the polygon so there should be 5 edges listed: *BG*, *BF*, *BC*, *AF*, *EF*
Choose any edge to start and label it 'I': *BG*(I), *BF*, *BC*, *AF*, *EF*
Unlabelled edges that cross *BG*: *BG*(I), *BF*, *BC*, *AF*(O), *EF*(O)
Unlabelled edges that cross *AF*, *EF:BG*(I), *BF*(I), *BC*, *AF*(O), *EF*(O)
Unlabelled edges that cross *BF*: *BG*(I), *BF*(I), *BC*(I), *AF*(O), *EF*(O)
All edges are labelled so the graph is planar:



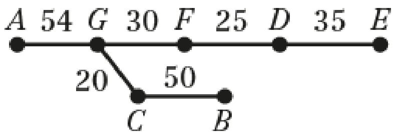
d



BG(I), *BF*, *BC*, *AF*, *EF*, *ED* and *EF* cross so graph is non-planar.

9 a *GC*, *FD*, *GF*, reject *CD*, *ED*, reject *EF*, *BC*, *AG*, reject *AB*.

b



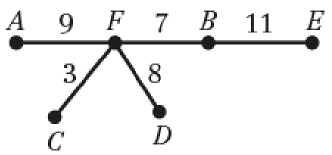
$$\text{cost} = (20+25+30+35+50+54)\times 1000 = £214\,000$$

10 a i Method:

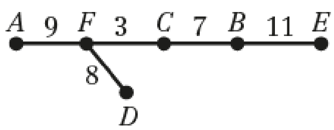
- Start at *A* and use this to start the tree.
- Choose the shortest edge that connects a vertex already in the tree to a vertex not yet in the tree. Add it to the tree.
- Continue adding edges until all vertices are in the tree.

$$AF,\ FC\begin{Bmatrix} FB \\ \text{or} \\ BC \end{Bmatrix}, FD,\ EB$$

ii Total length of spanning tree = 9+3+7+8+11 = 38



or



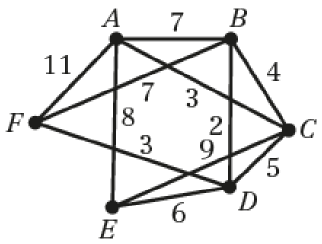
iii The tree is not unique, there are 2 of them (see above).

b i number of edges $= 7 - 1 = 6$

ii number of vertices $= n + 1$

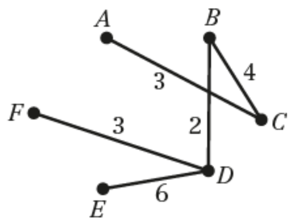
In a tree the number of edges is always one less than the number of nodes.

11 a



b $BD, \begin{Bmatrix} AC \\ DF \end{Bmatrix}, BC$, reject CD, DE. Length of tree = 2 + 3 + 3 + 4 + 6 = 18 km

c *DB*, *DF*, *BC*, *CA*, *DE*



length of tree is 18 km

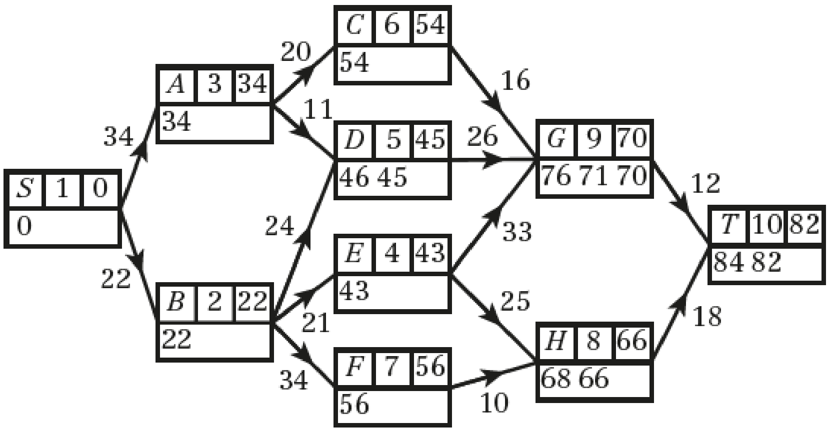
12 a In Prim's algorithm, the starting point can be any node, whereas Kruskal's algorithm starts from the arc of least weight. In Prim's algorithm, each new node and arc is added to the existing tree as it builds, whereas in applying Kruskal's algorithm, the arcs are selected according to their weight and may not be connected until the end.

b **i** Choose *G* to start the tree.
Add the arc of the least weight, *GH*, to the tree.
Consider arcs linking *G* to another vertex of least weight to the tree.
Continue to select an arc of least weight that joins a vertex already in the tree to a vertex not yet in the tree until all the vertices are connected: *GH, GI, HF, FD, DA, AB, AC, DE*

ii By inspection, order the arcs into ascending order of weight.
Select the arc of least weight to start the tree.
Consider the next arc of least weight, if it would form a cycle with the arcs already selected, reject it. Continue to select an arc of least weight until all vertices are connected: *GH*, *AB*, *AC*, *AD*, reject *BD*, *DF*, *GI*, reject *BC*, *FH*, reject *DG*, *DE*

c Weight is $6+7+8+9+10+10+11+15=76$

13 a



Route: $S-A-C-G-T$ length: 82

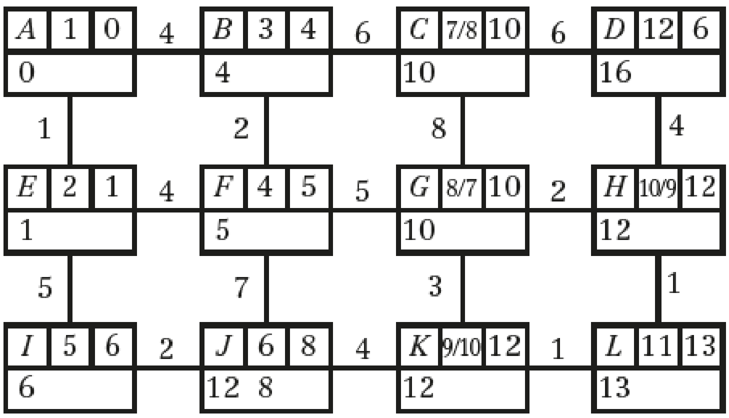
b For example

$82-12=70$ GT
$70-16=54$ CG
$54-20=34$ AC
$34-34=0$ SA

c Shortest route from S to $H + HT$

$S-B-F-H-T$ length: 84

14 a



For example;

$13-1=12$	HL	or	$13-1=12$	KL
$12-2=10$	GH		$12-4=8$	JK
$10-5=5$	FG		$8-2=6$	IJ
$5-4=1$	EF		$6-5=1$	EI
$1-1=0$	AE		$1-1=0$	AE

Shortest path is $\left\{\begin{matrix} A-E-F-G-H-L \\ A-E-I-J-K-L \end{matrix}\right\}$ length 13

b See the two equal length paths given above in part **a**.

15 a

Initial tables

	A	B	C	D
A	–	10	5	∞
B	10	–	3	7
C	5	3	–	11
D	1	7	11	–

	A	B	C	D
A	A	B	C	D
B	A	B	C	D
C	A	B	C	D
D	A	B	C	D

Initial distance table   Initial route table

∞: no direct route

1st iteration

	A	B	C	D
A	–	10	5	∞
B	10	–	3	7
C	5	3	–	11
D	1	7	6	–

	A	B	C	D
A	A	B	C	D
B	A	B	C	D
C	A	B	C	D
D	A	B	A	D

2nd iteration

	A	B	C	D
A	–	10	5	17
B	10	–	3	7
C	5	3	–	10
D	1	7	6	–

	A	B	C	D
A	A	B	C	B
B	A	B	C	D
C	A	B	C	B
D	A	B	A	D

3rd iteration

	A	B	C	D
A	–	8	5	15
B	8	–	3	7
C	5	3	–	10
D	1	7	6	–

	A	B	C	D
A	A	C	C	C
B	C	B	C	D
C	A	B	C	B
D	A	B	A	D

4th iteration

	A	B	C	D
A	–	8	5	15
B	8	–	3	7
C	5	3	–	10
D	1	7	6	–

	A	B	C	D
A	A	C	C	C
B	C	B	C	D
C	A	B	C	B
D	A	B	A	D

b From table, 15 miles

c The final route table shows that the shortest route from *A* to *D* is via *C*
The shortest route from *C* to *D* is via *B*. So, the shortest route from *A* to *D* is *ACBD*

16 $x = CD + DA = 7 + 10 = 17$
$y = AB + BC = 5 + 3 = 8$
$z = AB + BD = 5 + 6 = 11$

17 a i Shortest path through A is 18 + y or 26, both of which are greater than 17.

Shortest path through C is 23, which is greater than 17. So shortest path cannot go through A or C

ii Shortest path must go through B

$$S - B - D - T = 13 + x$$
$$13 + x = 17$$
$$x = 4$$

b If y = 0 shortest path is $S - A - D - T = 18$

If y = 5 shortest path is $S - C - D - T = 23$
so range is 18 to 23.

c For example, a person seeking the quickest route from home to work through a city. The arcs are the roads that may be chosen, the number the time, in minutes, to journey along that road. The nodes represent junctions.

18 a Odd vertices are B_1, B_2, E, G

$B_1B_2 + EG = 65 + 18 = 83$
$B_1E + B_2G = 41 + 42 = 83$
$B_1G + B_2E = 26 + 30 = 56$

Repeat B_1D, DG, B_2A, AE

Route: For example,

$$F - A - E - A - B_2 - A - C - E - F - G - D - H - G - D - B_1 - D - F$$

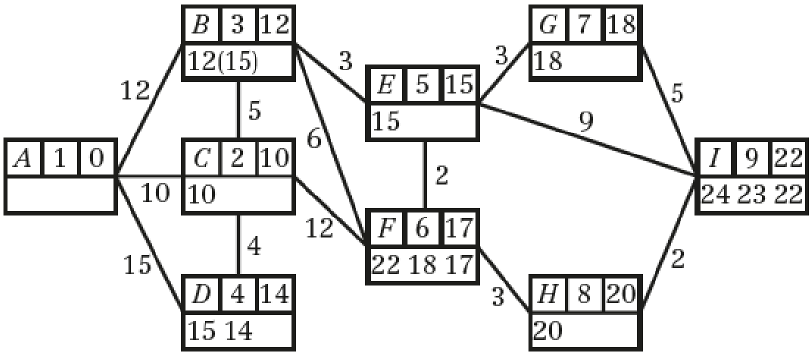
(All correct routes have 17 letters in their 'word')
length $= 129 + 56 = 185\,\text{km}$

b Now only the route between E and G needs repeating
so repeat $EF + FG = 18$
length of new route $= 129 + 18$
$= 147\,\text{km}$

19 a All arcs are to be traversed twice, this is, in effect, repeating each arc. So all valencies are even.

b e.g. $A - B - D - G - F - G - D - C - E - A - E - C - A - F - E - F - B - F - A - B - D - C - A$
(all correct routes will have 23 letters in their name)
length $= 2 \times 6 = 12\,\text{km}$

20 a



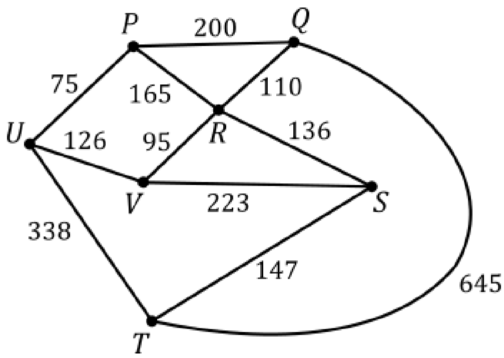
Shortest route is $A-B-E-F-H-I$, length 22 km

b i Odd vertices are *A* and *I* (only), so we need to repeat the shortest route from *A* to *I*. This was found in **a**.
So repeat *AB*, *BE*, *EF*, *FH*, *HI*.

ii For example $A-B-C-A-D-C-E-H-I-H-E-F-I-G-F-E-B-F-B-A$
(20 letters in route)

iii $91+22=113$ km

21



a Total length = 2260 m
Odd nodes *P*, *Q*, *S*, *T*, *U*, *V*
T and *P* remain odd.
QS + *UV* = 246 + 126 = 372 ← least weight
QU + *SV* = 275 + 223 = 498
QV + *SU* = 205 + 349 = 554
QS and *UV* gives the shortest pairing.
Roads to be traversed twice: *QR*, *RS*, *UV*

b Total of roads traversed twice $110+136+126=372$
Shortest route is $2260+372=2632$ m

22 a Odd valencies are at *A*, *B*, *C*, *D*, *F*, *G*
Route starts at *A* and finishes at *G* so these can remain odd.
Choose pairings of remaining odd vertices *B*, *C*, *D*, *F*
By inspection, these path lengths are:
BC + *DF* = 0.8 + 1.7 = 2.5
BD + *CF* = 1.3 + 2.3 = 3.6
BF + *CD* = 1.5 + 0.7 = 2.2 ← least weight
Repeating BF and *CD* minimises the total distance travelled.
Length $=9.5+2.2=11.7$ km

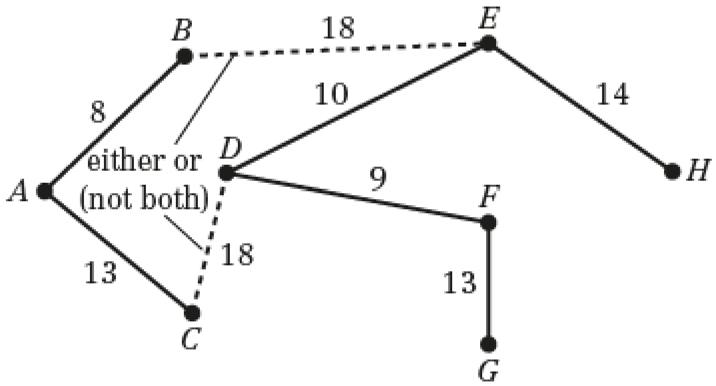
b $A-B-C-A-G-B-D-C-D-E-F-B-F-G$ (14 letters in route)

22 c Repeating *AC* and *BF* = 2.1
Minimum distance = 11.6 km
The engineer is correct. His new route is 0.1 km shorter.

23 a In the *practical* T.S.P each vertex must be visited *at least once*

In the *classical* T.S.P. each vertex must be visited *exactly once*

b $AB, DF, DE, (\text{reject } EF), \begin{Bmatrix} FG \\ AC \end{Bmatrix}, EH, \begin{Bmatrix} DC \\ \text{or} \\ BE \end{Bmatrix}$



c Initial upper bound $= 2 \times 85 = 170$ km

d When *CD* is part of tree

Use *GH* **(**saving 26**)** and BD (saving 19) giving a new upper bound of 125 km

Tour *A B D E H G F D C A*
e.g. when *BE* is part of tree

Use *CG* (saving 40) giving a new upper bound of 130 km

Tour *A B E H E D F G C A*

24 a

	A	*B*	*C*	*D*	*E*	*F*
A	–	20	30	32	12	15
B	20	–	10	25	32	16
C	30	10	–	15	35	19
D	32	25	15	–	20	34
E	12	32	35	20	–	16
F	15	16	19	34	16	–

Each row shows the shortest route.
The first row shows the shortest route starting at *A*. There are direct routes from *AB*, *AE* and *AF* and these are the shortest routes. *AC* (30) is by observation using *ABC* and *AD* (32) is by observation using *AED*.

b *AE* (12), *EF* (16), *FB* (16), *BC* (10), *CD* (15), *DA* (32)
101 km tour *AEFBCDA*

c In the original network *AD* is not a direct path. The tour becomes *A E F B C D E A*

24 d e.g.

$$\left.\begin{array}{l} B\ C\ D\ E\ A\ F\ B \\ C\ B\ F\ A\ E\ D\ C \\ D\ C\ B\ F\ A E\ D \\ E\ A\ F\ B\ C\ D\ E \\ F\ A\ E\ D\ C\ B\ F \end{array}\right\} \text{ length } 88$$

25 a i Minimum connector using Prim: *AC*, *CB*, *CD*, *CE*

length $= 98 + 74 + 82 + 103 = 357$ $\{1, 3, 2, 4, 5\}$

So upper bound $= 2 \times 357 = 714$

ii $A\underline{(98)}\ C\underline{(74)}\ A\underline{(131)}\ D\underline{(134)}\ E\underline{(115)}A$

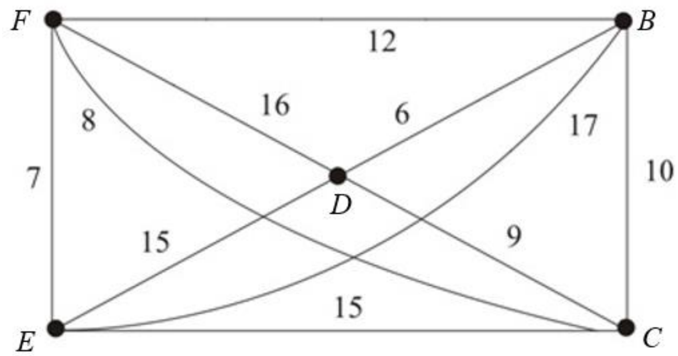
length $= 98 + 74 + 131 + 134 + 115 = 552$

b Residual minimum connector is *AC*, *CB*, *CD* length 254

Lower bound $= 254 + 103 + 115 = 472$

c $472 \leqslant \text{solution} \leqslant 552$

26 a Deleting vertex *A* we obtain



By Kruskal's algorithm an MST is *DB*(6), *EF*(7), *CF*(8), *DC*(9) of weight 30

The two edges of least weight at *A* are *AE*(7) and *AD*(8)

$\therefore$ A lower bound is 30 + 8 + 7 = 45

b i *A* – nearest neighbor *E*(7)

E – nearest neighbor *F*(7)

F – nearest neighbor *C*(8)

C – nearest neighbor *D*(9)

D – nearest neighbor *B*(6)

Complete tour with *BA*(12)

A E F C D B A – length 49

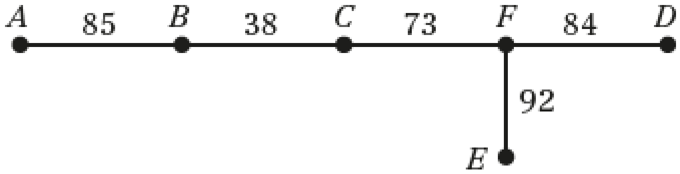
ii Choose a tour that does not use AB

e.g. *DB*(6) *BC*(10), *CF*(8), *FE*(4), *EA*(4)

Complete with *AD*(8), *D B C F E A D*.

Total weight 46

27 a Order of arcs: *AB*, *BC*, *CF*, *FD*, *FE*



b **i** $2 \times 372 = 744$

ii e.g. *AD* saves 105 giving 639
or *AE* saves 180 giving 564
AF saves 96 giving 648
DE saves 66 giving 678

c Residual M.S.T.
AB, *BC*, *AE*, *ED*

$$\text{Lower bound} = 341 + 73 + 84$$
$$= 498$$

28 a $AC(53),\ AD(54),\ DF(48),\ CE(58),\ EB(38)$

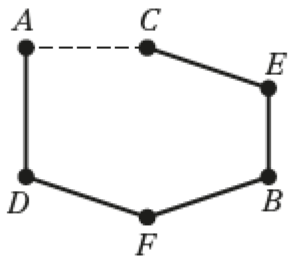
b **i** M.S.T. $XZ = 251 \times 2 = 502$

ii Finding a shortcut to below 360, e.g. *FB* =100 shortens by 151 so we get 251+100 = 351.

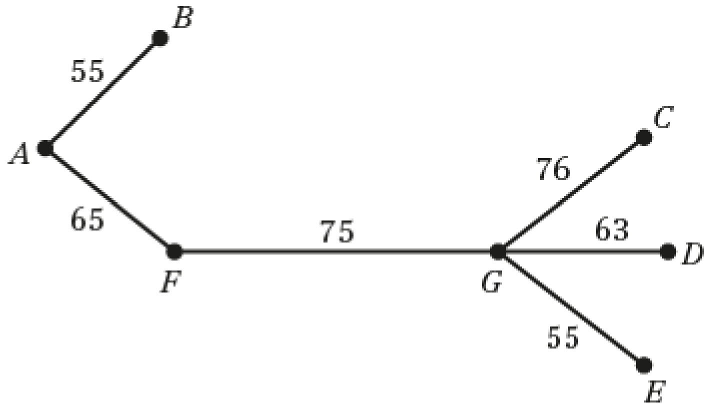
c M.S.T. is *DF*, *CE*, *EB*, *FB* length 244
The 2 shortest arcs are *AC* (53) and *AD* (54) giving a total of 351

d The optimal solution is 351 and is *A* – *C* – *E* – *B* – *F* – *D* – *A*

e



29 a Label column A, delete row A.
Scan all labelled columns and choose the least number.
Add that new vertex to the tree.
Label the new vertex's column and delete its row
Repeat the 3 steps until all vertices added.
Applying algorithm
order of vertex selection A, B, F, G, E, D, C



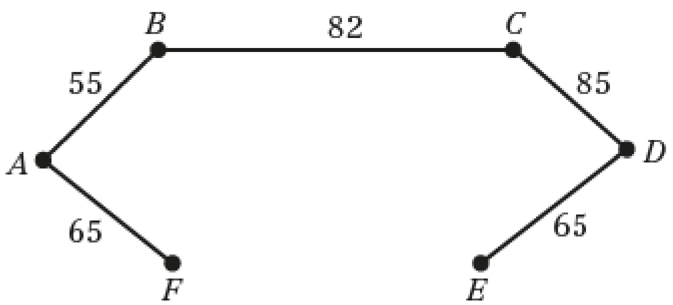
b Initial upper band $= 2 \times 389 = 778$ km

c Reducing upper bound by short cuts
e.g. Using *BC* = 82 instead of *BA* + *AF* + *FG* + *GC* leaves an upper bound of 589

Lists new route e.g. *A B C G D G E G F A*

States revisited vertices e.g. *G*

d



e Lower bound $= 352 + GD + GE$

$= 352 + 63 + 55$

$= 470$ km

f e.g. Use *GE* and *GF* (rather than *GD*)
length = 352 + 55 + 75 = 482 km

Route *A B C D E G F A*

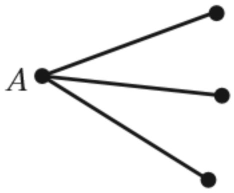
Challenge

1 a Let G be any finite simple graph with more than one vertex and with number of vertices $n \geqslant 2$
The maximal degree of any vertex in G is $\leqslant n-1$. Also, if our graph G is not connected, then the maximal degree is $< n-1$
Case 1: Assume that G is connected. We cannot have a vertex of degree 0 in G, so the set of vertex degrees is a subset of $S = \{1, 2, ..., n-1\}$. Since the graph G has n vertices and there are $n-1$ possible degree options, there must be two vertices of the same degree in G.
Case 2: Assume that G is not connected. G has no vertex of degree $n-1$, so the set of vertex degrees is a subset of $S' = \{1, 2, ..., n-2\}$. Again, we have n vertices and $n-1$ possible degree options, so there must be two vertices of the same degree in G.

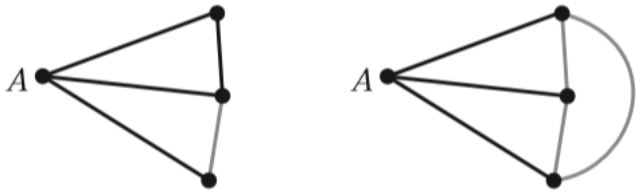
b i By inspection, possible sets are:
Blue: *ABD*, *ACD*
Red: *BCF*, *DEF*

ii For K_6 any vertex will have a valency of 5, an edge to each of the other points.

With 5 lines there must be at least three of one colour so there are four points connected by the same colour.



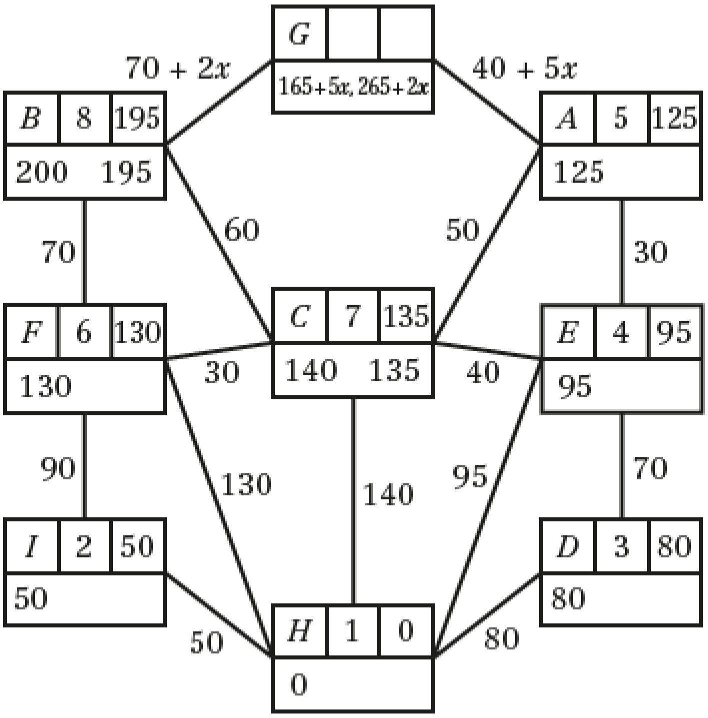
For the three lines connecting the new points, if one is the original colour then a set of three vertices is made with the original colour. If none are the original colour then the three vertices make a set of three themselves.



iii

Challenge

2 a



Via A: $H-E-A-G$ length $165+5x$
Via B: $H-E-C-B-G$ length $265+2x$

b $165+5x=265+2x \Rightarrow x=33\frac{1}{3}$

So range is $0 \leqslant x \leqslant 33\frac{1}{3}$

3 a $9\frac{1}{2}x-26$

b The only vertices of odd order are B and C, we have to repeat the shortest path between B and C.
If $x \geqslant 9$ the shortest path is BC (direct)
Weight of network + BC = 100
$\left(9\frac{1}{2}x-26\right)+x=100 \Rightarrow x=12$
If $x<9$ the shortest path is BAC of length $2x-9$
$\left(9\frac{1}{2}x-26\right)+2x-9=100 \Rightarrow x=11\frac{17}{23} \geqslant 9$
so inconsistent and hence $BA+AC \geqslant BC$

Challenge

3 c The only vertices of odd order are *B* and *C*, we have to repeat the shortest path between *B* and *C*.

If $x \geqslant 9$ the shortest path is *BC* (direct).

Weight of network + *BC* = 100

$$(9\frac{1}{2}x - 26) + x = 100 \Rightarrow x = 12$$

If $x < 9$ the shortest path is *BAC* of length $2x - 9$

$(9\frac{1}{2}x - 26) + 2x - 9 = 100 \Rightarrow x = 11\frac{17}{23} \geqslant 9$ so inconsistent and hence $x = 12$

4 a Minimum spanning tree $= 751$
Initial upper bound $= 2 \times 751 = 1502$
Taking shortcut *AH* saves $120 + 131 - 144 = 107$
Tour length $= 1395$
Tour route: *ABACAEHFGDGFHA*

b Delete *G* and use Prim's algorithm starting at *A*
RMST $= 672$
Lower bound by deleting *G* = 672 + 144 + 155 = 971
Route is not a tour, so does not give an optimal solution.