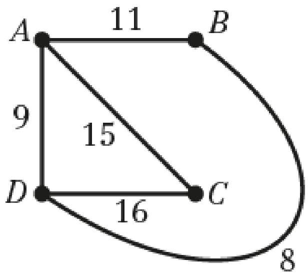
Graphs and networks Mixed exercise

1 E.g.



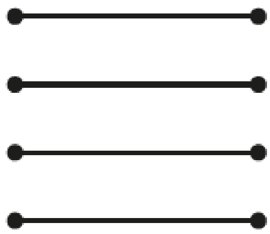
2 **a**, **e** and **h** are isomorphic.

b and **i** are isomorphic.

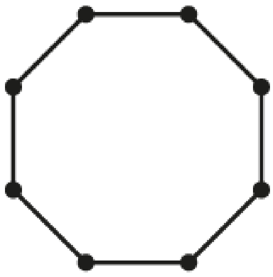
c and **g** are isomorphic.

d and **f** are isomorphic.

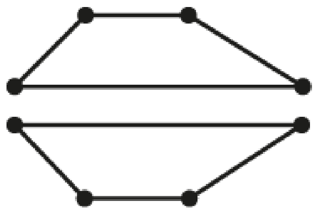
3 **a**



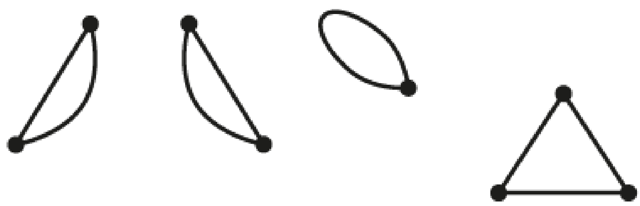
b **i**



ii



iii

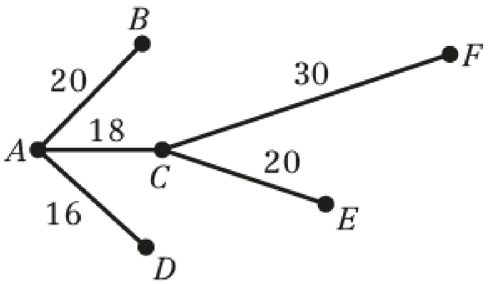


4 a A distance matrix gives the lengths of edges between pairs of vertices, whereas an adjacency matrix gives the number of edges between pairs of vertices.

b

	A	B	C	D	E	F
A	-	20	18	16	-	-
B	20	-	15	-	-	50
C	18	15	-	10	20	30
D	16	-	10	-	23	-
E	-	-	20	23	-	25
F	-	50	30	-	25	-

4 c E.g.



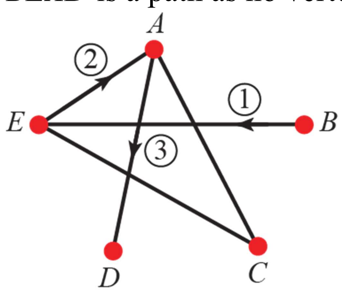
Weight $= 20 + 16 + 18 + 20 + 30 = 104$

5 There are exactly $v-1$ edges. To see that, imagine a process of drawing the spanning tree. After drawing the first arc, only 2 vertices are connected. Then, each time we draw an arc, we attach to the connected component exactly one vertex. Hence, to connect the entire graph we need $v-1$ edges.

6 If we start with *PQ* we have *PQR* and *PQTR;* if we start with *PT* the only option is *PTR*; starting with *PS* leads to *PSR.*

7 a There are 3 edges incident to *A*, so the valency of *A* is 3.

b i *BEAD* is a path as no vertex is visited more than once.



Note: *BECAD* is also a possible path.

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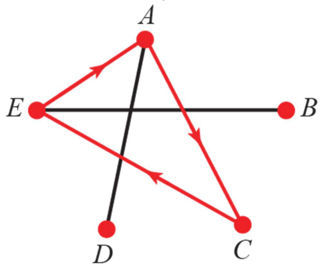
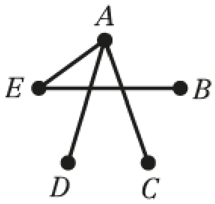
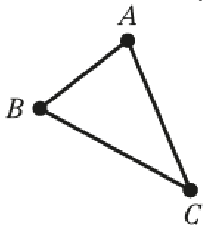
7 b ii $ACEA$ is a cycle.



c i It suffices to remove an edge belonging to cycle $ACEA$ to obtain a spanning tree, e.g.



ii Vertices A, C, E joined by respective edges form a complete graph K_3



8 a In the required graph, there would be 3 nodes of odd degree. It contradicts the consequence of Euler's handshaking lemma which says that the number of odd nodes is even.

Alternatively, we can try to draw the graph and reach some contradiction.

b By Euler's handshaking lemma:
Sum of degrees $= 2 \times$ Number of edges
$\Rightarrow k^2 - 3k + k + 1 + 8 - k + k - 4 = 2 \times 10$
$\Rightarrow k^2 - 2k - 15 = 0$
$\Rightarrow (k-5)(k+3) = 0$
$\Rightarrow k = 5$ or $k = -3$
For $k = 5$ the degrees are $10, 6, 3, 1$; for $k = -3$ the degrees are $18, -2, 11, -7$
Negative degree is not possible, so $k = 5$

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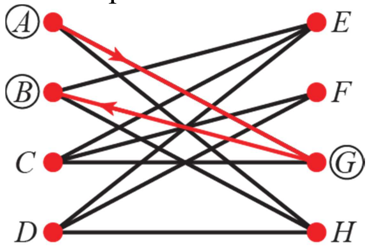
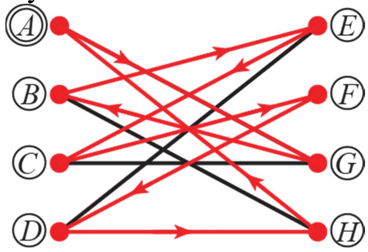
9 a It is helpful to encircle the vertices which have been visited.

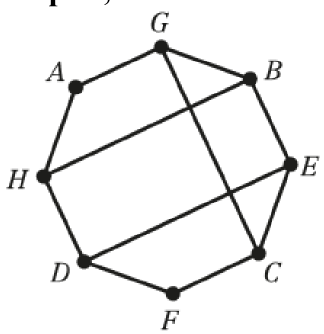


By trial and error we can complete the Hamiltonian cycle *AGBECFDHA*

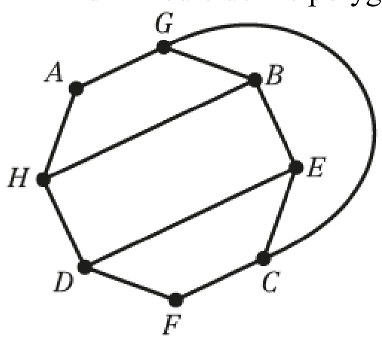


b **Step 1:** Use the Hamiltonian cycle *AGBECFDHA* identified in part **a.**

Steps 2, 3: Draw an octagon (preferably regular) based on that cycle and add respective edges.



Steps 4, 5: Choose edge *CG,* which is inside the octagon and crosses two other edges. Move edges *BH* and *DE* outside the polygon. Go to **Step 4**.



Step 4: There are no edges crossing each other inside the polygon, so we conclude that the graph is planar.

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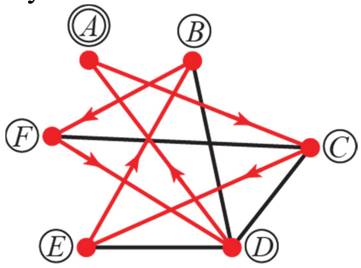
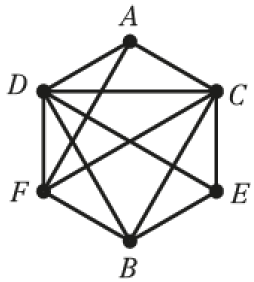
10 a By trial and error we can find, e.g.

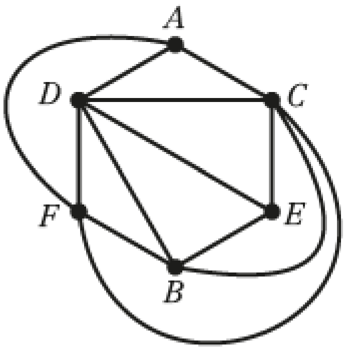


b **Step 1:** Use the Hamiltonian cycle *ACEBFDA* identified in part **a.**

Steps 2, 3: Draw a hexagon (preferably regular) with vertices matching the cycle and add respective edges, so that the graph is isomorphic to the original one.



Steps 4, 5: Fix *DE* and move *AF* and *CF* outside the hexagon. Go to **Step 4**.



Step 4: There are no more edges crossing each other inside the polygon, so we conclude that the graph is planar.

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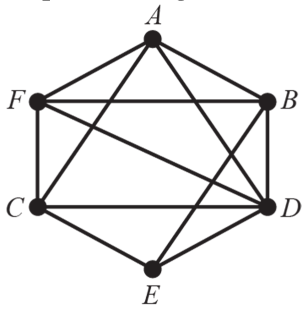
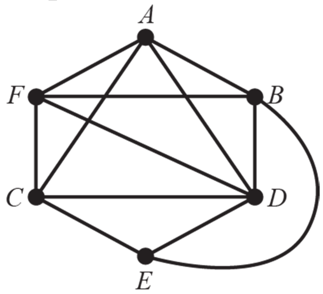
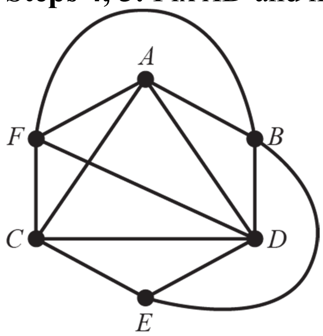
10 c **Steps 2, 3:** Draw a hexagon (preferably regular) with vertices matching cycle *ABDECFA* and add respective edges.



Steps 4, 5: Fix *CD* and move *BE* outside the hexagon. Go to **Step 4**.



Steps 4, 5: Fix *AD* and move *BF* outside. Go to **Step 4**.



Steps 4, 5: Choose edge *AC.* It is not possible to move *DF* outside without crossing other edges, so go to **Step 6**.

Step 6: Choose edge *DF* and go to **Step 5**.

Step 5: Edge *AC* cannot be moved outside the hexagon without crossing another edge so go to **Step 6**.

Step 6. There are no more edges inside the hexagon which we can choose, so we conclude the graph is not planar.

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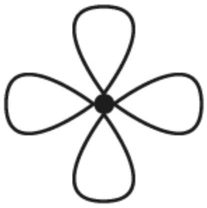
Challenge

a $V = 7$, $R = 7$, $E = 13 \Rightarrow V + R - E = 7 + 7 - 13 = 1$

b For a graph with one vertex, all edges must be loops. For example:



$V = 1$, $R = 4$, $E = 4 \Rightarrow V + R - E = 1 + 4 - 4 = 1$

c Observe that for a planar graphs with one vertex ($V = 1$) $E = R$
Hence, $V + R - E = V + (R - E) = V = 1$

d In G', we have $V' = V - 1$, $R' = R$ and $E' = E - 1$
$\Rightarrow V' + R' - E' = (V - 1) + R - (E - 1) = V + R - E = 1$

e We will prove the claim by induction on the number of vertices.

First, note that we have verified the formula for $V = 1$ in part **b**.

Suppose the formula holds for all planar connected graphs with V vertices. We will demonstrate that it also holds for all planar connected graphs with $V + 1$ vertices. Given a planar connected graph G with $V + 1$ vertices, we contract it to G' with V vertices. Contraction is possible as the graph is connected, so it contains at least one edge. The formula holds for graph G' by the inductive assumption. Moreover, observe that the argument from **d** works also in reverse, i.e. $V' + R' - E' = 1$ implies $V + R - E = 1$

Thus, by mathematical induction the formula holds for all planar connected graphs.

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