Quality of tests Mixed exercise 8

1 a $H_0: p = 0.35 \quad H_1: p > 0.35$

Assume H_0, so that $X \sim B(15, 0.35)$

Significance level 5%, so require c such that $P(X \geqslant c) < 0.05$

From the binomial cumulative distribution tables

$P(X \geqslant 8) = 1 - P(X \leqslant 7) = 1 - 0.8868 = 0.1132$

$P(X \geqslant 9) = 1 - P(X \leqslant 8) = 1 - 0.9578 = 0.0422$

$P(X \geqslant 8) > 0.05$ and $P(X \geqslant 9) < 0.05$ so the critical value is 9

Hence the critical region is $X \geqslant 9$

b $\text{Size} = P(\text{Type I error}) = P(X \geqslant 9 \mid p = 0.35) = 0.0422$

c $\text{Power} = P(H_0 \text{ is rejected} \mid p = 0.5) = P(X \geqslant 9 \mid p = 0.5)$

$= 1 - P(X \leqslant 8 \mid p = 0.5)$

$= 1 - 0.6964 = 0.3036$

2 a $H_0: \lambda = 3.5 \quad H_1: \lambda < 3.5$

Assume H_0, so that $X \sim Po(3.5)$

Significance level 5%, so require c such that $P(X \leqslant c) < 0.05$

From the Poisson cumulative distribution tables

$P(X \leqslant 1) = 0.1359$ and $P(X = 0) = 0.0302$

$P(X \leqslant 1) > 0.05$ and $P(X = 0) < 0.05$ so the critical value is 0

Hence the critical region is $X = 0$

b $\text{Size} = P(\text{Type I error}) = P(X = 0 \mid \lambda = 0.35) = 0.0302$

c $\text{Power} = P(H_0 \text{ is rejected} \mid \lambda = 3.0) = P(X = 0 \mid \lambda = 3.0) = 0.0498$

3 a $H_0: \mu = 8 \quad H_1: \mu \neq 8$

Assume H_0, so that $\bar{X} \sim N\left(8, \frac{9}{18}\right)$

Standardise the $\bar{X}$ variable

$$Z = \frac{\bar{X} - 8}{\frac{3}{\sqrt{18}}} = \sqrt{2}(\bar{X} - 8)$$

Significance level 5%, so require 2.5% in each tail

From the tables, the critical region for Z is $Z > 1.96$ or $Z < -1.96$

So the critical values for $\bar{X}$ are given by

$\sqrt{2}(\bar{X} - 8) = \pm 1.96$

$\Rightarrow \bar{X} = 6.6141$ and $\bar{X} = 9.3859$ (answers to 4 d.p.)

So the critical region for $\bar{X}$ is $\bar{X} < 6.6141$ or $\bar{X} > 9.3859$ (answers to 4 d.p.)

b $P(\text{Type I error}) = \text{significance level} = 0.05$

3 c Using the normal cumulative distribution function on a calculator:

$$\text{P(Type II error)} = \text{P}(6.6141 \leqslant \bar{X} \leqslant 9.3859 \mid \mu = 7)$$
$$= \text{P}(\bar{X} \leqslant 9.3859 \mid \mu = 7) - \text{P}(\bar{X} \leqslant 6.6141 \mid \mu = 7)$$
$$= 0.9996 - 0.2926 = 0.7070 \text{ (4 d.p.)}$$

d Power $= 1 - \text{P(Type II error)} = 1 - 0.7070 = 0.2930$

4 a Let the random variable X denote the number of geese observed flying in a migratory pattern in a day, then the null hypothesis H_0 is $\lambda = 10$ and $X \sim \text{Po}(10)$

This hypothesis is only rejected if they observe 3 or fewer geese on the given day and then 2 or fewer geese on the next day; or if they observe 18 or more geese on the given day and then 19 or more geese on the next day.

So Size $= \text{P}(H_0 \text{ rejected} \mid H_0 \text{ true})$

$$= \text{P}(X \leqslant 3) \times \text{P}(X \leqslant 2) + \text{P}(X \geqslant 18) \times \text{P}(X \geqslant 19)$$
$$= 0.010\,336\ldots \times 0.002\,769\ldots + 0.014\,278\ldots \times 0.007\,187\ldots$$
$$= 0.00\,013\,12 \text{ (7 d.p.)}$$

b Power $= \text{P}(H_0 \text{ rejected} \mid \lambda = 5)$

$$= \text{P}(X \leqslant 3) \times \text{P}(X \leqslant 2) + \text{P}(X \geqslant 18) \times \text{P}(X \geqslant 19)$$
$$= 0.265\,026\ldots \times 0.124\,652\ldots + 0.000\,005\ldots \times 0.000\,001\ldots$$
$$= 0.0330 \text{ (4 d.p.)}$$

5 a $H_0 : \lambda = 4.5 \quad H_1 : \lambda > 4.5$

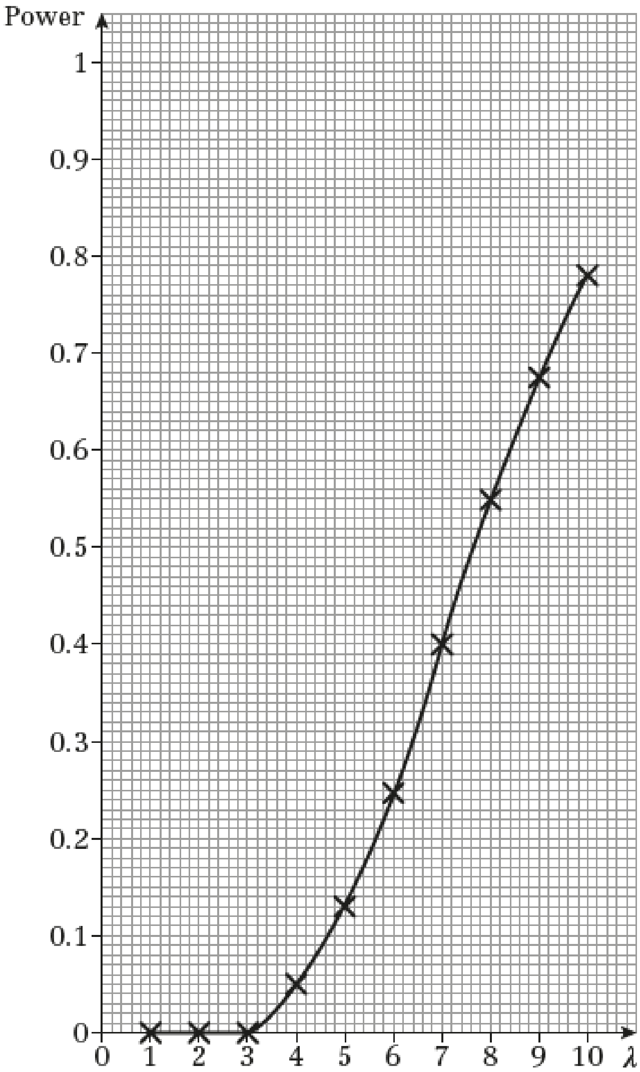
Critical region $X \geqslant 8$

$$\text{Size} = \text{P}(X \geqslant 8 \mid \lambda = 4.5) = 1 - \text{P}(X \leqslant 7 \mid \lambda = 4.5)$$
$$= 1 - 0.9134 = 0.0866$$

b i Power $= \text{P}(X \geqslant 8 \mid X \sim \text{Po}(\lambda)) = 1 - \text{P}(X \leqslant 7 \mid X \sim \text{Po}(\lambda))$

$$\lambda = 4 \Rightarrow r = 1 - 0.9489 = 0.0511$$
$$\lambda = 6 \Rightarrow s = 1 - 0.7440 = 0.2560$$
$$\lambda = 9 \Rightarrow t = 1 - 0.3239 = 0.6761$$

5 b ii



6 a $H_0 : p = 0.45 \quad H_1 : p < 0.45$

Critical region $X \leqslant 3$

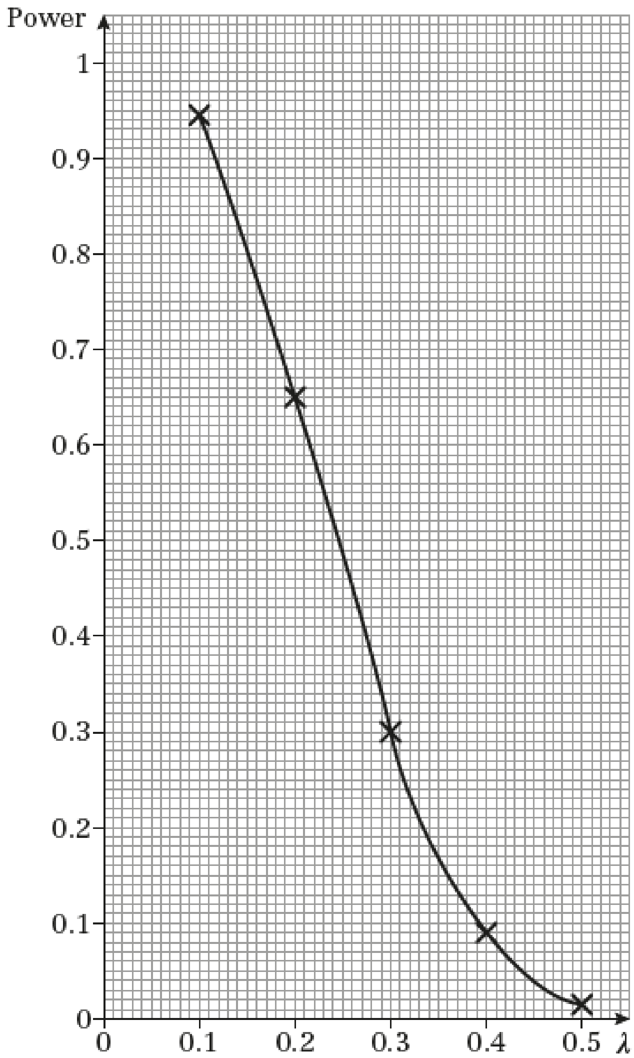
$\text{Size} = P(X \leqslant 3 \mid X \sim B(15, 0.45)) = 0.0424$

b $\text{Power} = P(X \leqslant 3 \mid X \sim B(15, p))$

$p = 0.2 \Rightarrow s = 0.6482$

$p = 0.4 \Rightarrow t = 0.0905$

6 c



7 a $H_0 : \lambda = 2$ $\quad$ $H_1 : \lambda > 2$

(Quality the same) $\quad$ (Quality is poorer)

b Assume H_0, so that X $\quad$ Po(2)

Require c such that $P(X \geqslant c) \approx 0.05$

From the Poisson cumulative distribution tables

$P(X \geqslant 4) = 1 - P(X \leqslant 3) = 1 - 0.8571 = 0.1429$

$P(X \geqslant 5) = 1 - P(X \leqslant 4) = 1 - 0.9473 = 0.0527$

$P(X \geqslant 6) = 1 - P(X \leqslant 5) = 1 - 0.9834 = 0.0166$

$P(X \geqslant 5)$ is closest to 0.05 so the critical value is 5

Hence the critical region is $X \geqslant 5$

c Power = P(H_0 is rejected | X $\quad$ Po(4))

$= P(X \geqslant 5 \mid \lambda = 4) = 1 - P(X \leqslant 4 \mid \lambda = 4)$

$= 1 - 0.6288 = 0.3712$

8 a $H_0 : \lambda = 2$ $H_1 : \lambda > 2$
(as good) (worse)

b Assume H_0, so that $X \sim Po(2)$
Require c such that $P(X \geqslant c) \approx 0.05$
From the Poisson cumulative distribution tables
$P(X \geqslant 5) = 1 - P(X \leqslant 4) = 1 - 0.9473 = 0.0527$
$P(X \geqslant 6) = 1 - P(X \leqslant 5) = 1 - 0.9834 = 0.0166$
$P(X \geqslant 5)$ is closest to 0.05 so the critical value is 5
Hence the critical region is $X \geqslant 5$

c Power $= P(H_0 \text{ is rejected} \mid X \sim Po(3))$
$= P(X \geqslant 5 \mid \lambda = 3) = 1 - P(X \leqslant 4 \mid \lambda = 3)$
$= 1 - 0.8153 = 0.1847$

d Let the random variable Y denote the number of faulty garments produced by the machinist over the three days
$H_0 : \lambda = 6$ $H_1 : \lambda > 6$
Assume H_0, so that $Y \sim Po(6)$
Require c such that $P(Y \geqslant c) \approx 0.05$
From the Poisson cumulative distribution tables
$P(Y \geqslant 10) = 1 - P(Y \leqslant 9) = 1 - 0.9161 = 0.0839$
$P(Y \geqslant 11) = 1 - P(Y \leqslant 10) = 1 - 0.9574 = 0.0426$
$P(Y \geqslant 11)$ is closest to 0.05 so the critical value is 11
Hence the critical region is $Y \geqslant 11$

e Power $= P(H_0 \text{ is rejected} \mid Y \sim Po(9))$ (3 days has mean $= 3 \times 3 = 9$)
$= P(Y \geqslant 11 \mid \lambda = 9) = 1 - P(X \leqslant 10 \mid \lambda = 9)$
$= 1 - 0.7060 = 0.2940$

f Second test is more powerful as monitors the performance of the machinist over more days.

9 a $H_0: \mu = 6 \quad H_1: \mu < 6$

Critical region is $X \leqslant 2$

Size = P(Type I error) = $P(X \leqslant 2 \mid \mu = 6) = 0.0620$

b Power function $= P(X \leqslant 2 \mid X \sim \text{Po}(\mu))$

$$= P(X = 0 \mid X \sim \text{Po}(\mu)) + P(X = 1 \mid X \sim \text{Po}(\mu)) + P(X = 2 \mid X \sim \text{Po}(\mu))$$

$$= \frac{e^{-\mu}\mu^0}{0!} + \frac{e^{-\mu}\mu^1}{1!} + \frac{e^{-\mu}\mu^2}{2!}$$

$$= e^{-\mu} + e^{-\mu}\mu + \frac{1}{2}e^{-\mu}\mu^2$$

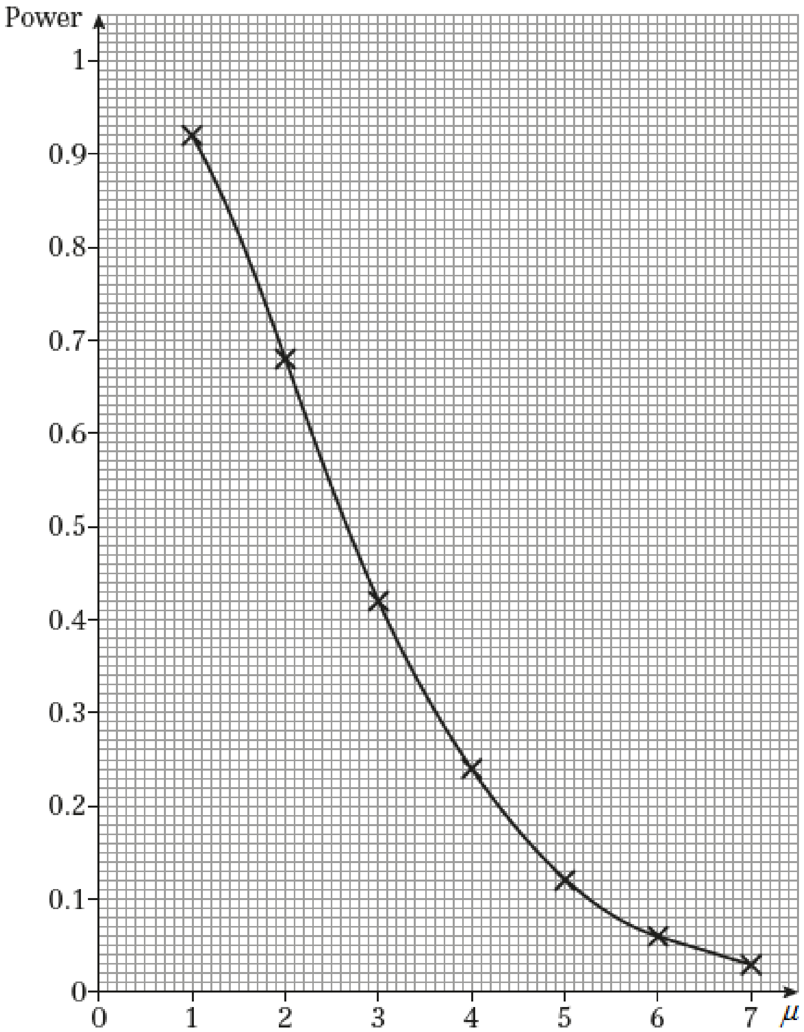
$$= e^{-\mu}\left(1 + \mu + \frac{\mu^2}{2}\right)$$

$$= \frac{1}{2}e^{-\mu}(2 + 2\mu + \mu^2)$$

c $s = \frac{1}{2}e^{-2}(2 + 4 + 4) = 5e^{-2} = 0.6767$ (4 d.p.)

$t = \frac{1}{2}e^{-5}(2 + 10 + 25) = 18.5e^{-5} = 0.1247$ (4 d.p.)

d



9 e Reading from the graph, at the point that the graph intersects the line *Power* = 0.8, $\mu \approx 1.55$ so the power of this test is greater than 0.8 for $\mu < 1.55$

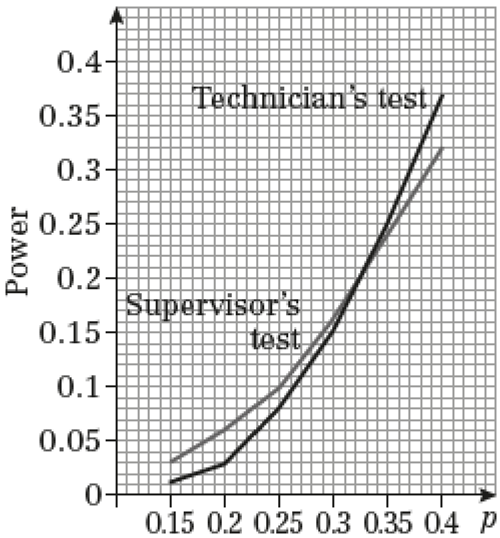
10 a $H_0 : p = 0.1 \quad H_1 : p > 0.1 \quad X \sim B(10, p)$
Critical region is $X \geqslant 5$
Size = P(Type I error) = $P(X \geqslant 5 \mid p = 0.1)$
$= 1 - P(X \leqslant 4 \mid p = 0.1) = 1 - 0.9984 = 0.0016$

b $u = P(H_0 \text{ rejected} \mid p = 0.25)$
$= 1 - P(X \leqslant 4 \mid p = 0.25) = 1 - 0.9219 = 0.0781$
$= 0.08$ (2 d.p.)

c $H_0 : p = 0.1 \quad H_1 : p > 0.1 \quad Y \sim B(5, p)$
Critical region is $Y \geqslant 3$
Size = P(Type I error) = $P(Y \geqslant 3 \mid p = 0.1)$
$= 1 - P(Y \leqslant 2 \mid p = 0.1) = 1 - 0.9914 = 0.0086$

d $v = P(H_0 \text{ rejected} \mid p = 0.35)$
$= 1 - P(Y \leqslant 2 \mid p = 0.35) = 1 - 0.7648 = 0.2352$
$= 0.24$ (2 d.p.)

e



f i 0.325
ii With *p* greater than this value, the technician's test is stronger than the supervisor's.

g The test is more powerful for probabilities closer to zero (< 0.325), and it is quicker to check 5 items for defects than to test 10 items.

11 a $H_0 : \lambda = 0.3 \quad H_1 : \lambda > 0.3 \quad X \sim Po(10\lambda)$
Critical region is $X \geqslant 6$
Size = P(Type I error) = $P(X \geqslant 6 \mid X \sim Po(3))$
$= 1 - P(X \leqslant 5 \mid X \sim Po(3)) = 1 - 0.9161 = 0.0839$

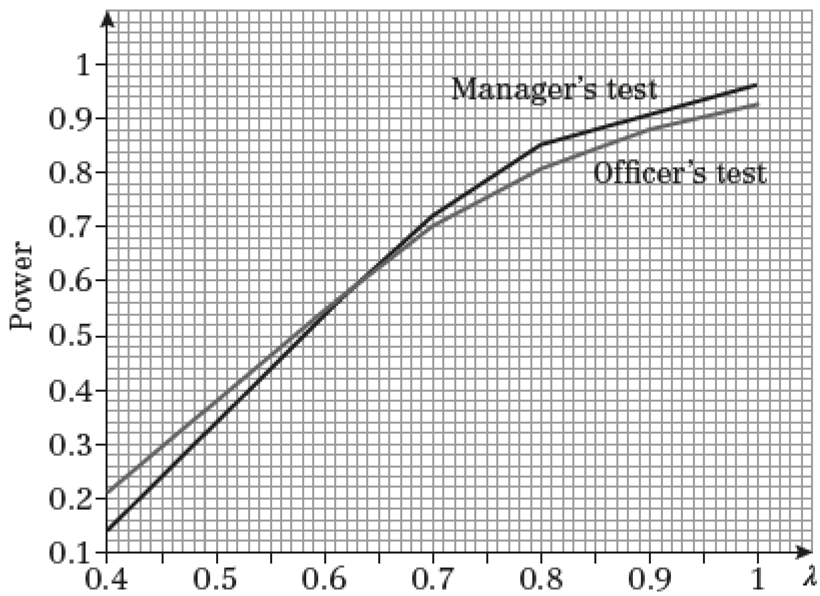
11 b $a = \text{P}(\text{H}_0 \text{ rejected} \mid X \sim \text{Po}(5))$
$= 1 - \text{P}(X \leqslant 5 \mid X \sim \text{Po}(5)) = 1 - 0.6160 = 0.3840$
$= 0.38$ (2 d.p.)

c $\text{H}_0 : \lambda = 0.3 \quad \text{H}_1 : \lambda > 0.3$
Assume H_0, so that $X \sim \text{Po}(15 \times 0.3)$, i.e. $X \sim \text{Po}(4.5)$
Significance level 5%, so require c such that $\text{P}(X \geqslant c) < 0.05$
From the Poisson cumulative distribution tables
$\text{P}(X \geqslant 8) = 1 - \text{P}(X \leqslant 7) = 1 - 0.9134 = 0.0866$
$\text{P}(X \geqslant 9) = 1 - \text{P}(X \leqslant 8) = 1 - 0.9597 = 0.0403$
$\text{P}(X \geqslant 8) > 0.05$ and $\text{P}(X \geqslant 9) < 0.05$ so the critical value is 9
Hence the critical region is $X \geqslant 9$

d Size $= \text{P}(\text{Type I error}) = \text{P}(X \geqslant 9 \mid X \sim \text{Po}(4.5))$
$= 1 - \text{P}(X \leqslant 8 \mid X \sim \text{Po}(4.5)) = 1 - 0.9597 = 0.0403$

e $b = \text{P}(\text{H}_0 \text{ rejected} \mid X \sim \text{Po}(13.5))$
$= 1 - \text{P}(X \leqslant 8 \mid X \sim \text{Po}(13.5)) = 1 - 0.0790 = 0.9210$
$= 0.92$ (2 d.p.)

f



g i 0.63
ii With λ greater than this value, the manager's test is more powerful.

Challenge

a Let the random variable X be the number of times a pair of ones appears in 12 rolls of the dice, then $X \sim \mathrm{B}(15, p)$

$\mathrm{H}_0 : \lambda = \frac{1}{36}$ $\quad \mathrm{H}_1 : \lambda > \frac{1}{36}$ $\quad X \sim \mathrm{B}\left(12, \frac{1}{36}\right)$

Critical region $X \geqslant 2$

$$\text{Size} = \mathrm{P}\left(X \geqslant 2 \mid X \sim \mathrm{B}\left(12, \frac{1}{36}\right)\right) = 1 - \mathrm{P}\left(X \leqslant 1 \mid X \sim \mathrm{B}\left(12, \frac{1}{36}\right)\right)$$
$$= 1 - 0.9577 = 0.0423 \text{ (4 d.p.)}$$

b
$$\text{Power} = \mathrm{P}(X \geqslant 2 \mid X \sim \mathrm{B}(12, p)) = 1 - \mathrm{P}(X \leqslant 1 \mid X \sim \mathrm{B}(12, p))$$
$$= 1 - \mathrm{P}(X = 0 \mid X \sim \mathrm{B}(12, p)) - \mathrm{P}(X = 1 \mid X \sim \mathrm{B}(12, p))$$
$$= 1 - (1-p)^{12} - 12p(1-p)^{11}$$

c
$$\text{Size} = \mathrm{P}(\text{Type I error}) = \mathrm{P}\left(X \geqslant 4 \mid X \sim \mathrm{B}\left(6, \frac{1}{6}\right)\right) + \mathrm{P}\left(X \leqslant 3 \mid X \sim \mathrm{B}\left(6, \frac{1}{6}\right)\right)\mathrm{P}\left(X \geqslant 4 \mid X \sim \mathrm{B}\left(6, \frac{1}{6}\right)\right)$$
$$= 0.008702\ldots + (0.991298\ldots \times 0.008702\ldots) = 0.0173 \text{ (4 d.p.)}$$

d The probability p of a double 1 in this situation is $q \times \frac{1}{6}$

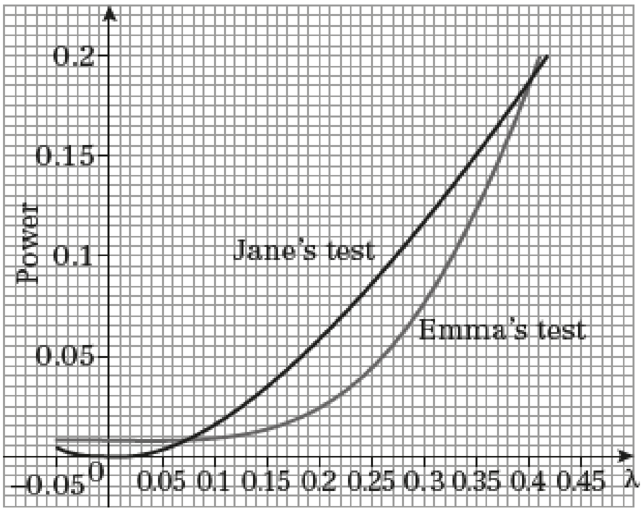
Substituting for $p = q \times \frac{1}{6}$ in the equation in part **b** gives

$$\text{Power} = 1 - \left(1 - \frac{q}{6}\right)^{12} - 12 \times \frac{q}{6}\left(1 - \frac{q}{6}\right)^{11}$$
$$= 1 - \left(1 - \frac{q}{6}\right)^{12} - 2q\left(1 - \frac{q}{6}\right)^{11}$$

e
$$\text{Power} = \mathrm{P}\left(X \geqslant 4 \mid X \sim \mathrm{B}\left(6, \frac{1}{6}\right)\right) + \mathrm{P}\left(X \leqslant 3 \mid X \sim \mathrm{B}\left(6, \frac{1}{6}\right)\right)\mathrm{P}\left(X \geqslant 4 \mid X \sim \mathrm{B}(6, q)\right)$$
$$= 0.0087 + 0.9913 \times \left(15q^4(1-q)^2 + 6q^5(1-q) + q^6\right)$$
$$= 0.0087 + 0.9913 \times \left(15q^4 - 30q^5 + 15q^6 + 6q^5 - 6q^6 + q^6\right)$$
$$= 0.0087 + 0.9913 \times \left(15q^4 - 24q^5 + 10q^6\right)$$
$$= 0.0087 + 14.8695q^4 - 23.7912q^5 + 9.913q^6$$

Challenge

f



g As the power of Jane's test is greater than that of Emma's when $0.1 < q < 0.4$, recommend that Jane's test is used.