Quality of tests 8C

1 a $H_0 : \mu = 20 \quad H_1 : \mu > 20$

Assume H_0, so that $\bar{X} \sim N\left(20, \frac{3^2}{25}\right)$

Standardise the $\bar{X}$ variable

$Z = \frac{\bar{X} - 20}{\frac{3}{5}} = \frac{5(\bar{X} - 20)}{3}$

Significance level 5%

From the tables, the 5% critical region for Z is $Z > 1.6449$

So the critical region for $\bar{X}$ is given by

$\frac{5(\bar{X} - 20)}{3} > 1.6449 \Rightarrow \bar{X} > 20.9869\ldots$

b Power $= 1 - \text{P(Type II error)}$

$= 1 - P(\bar{X} \leqslant 20.9869... \mid \mu = 20.8) = 1 - 0.6223 = 0.3777$ (4 d.p.)

2 a $H_0 : p = 0.35 \quad H_1 : p > 0.35$

Assume H_0, so that $X \sim B(20, 0.35)$

Significance level 5%, so require c such that $P(X \geqslant c) < 0.05$

From the binomial cumulative distribution tables

$P(X \geqslant 11) = 1 - P(X \leqslant 10) = 1 - 0.9468 = 0.0532$

$P(X \geqslant 12) = 1 - P(X \leqslant 11) = 1 - 0.9804 = 0.0196$

$P(X \geqslant 11) > 0.05$ and $P(X \geqslant 12) < 0.05$ so the critical value is 12

Hence the critical region is $X \geqslant 12$

Size $= \text{P(Type I error)} = P(X \geqslant 12 \mid p = 0.35) = 0.0196$

b Find the answer using a calculator as the tables do not give cumulative probabilities of the binomial distribution for $p = 0.36$

Power $= 1 - \text{P(Type II error)} = P(\bar{X} \geqslant 12 \mid p = 0.36)$

$= 1 - P(\bar{X} \leqslant 11 \mid p = 0.36) = 1 - 0.9753 = 0.0247$ (4 d.p.)

3 a $H_0 : \lambda = 4.5 \quad H_1 : \lambda < 4.5$

Assume H_0, so that $X \sim Po(4.5)$

Significance level 5%, so require c such that $P(X \leqslant c) < 0.05$

From the Poisson cumulative distribution tables

$P(X \leqslant 1) = 0.0611$ and $P(X = 0) = 0.0111$

$P(X \leqslant 1) > 0.05$ and $P(X = 0) < 0.05$ so the critical value is 0

Hence the critical region is $X = 0$

Size $= \text{P(Type I error)} = P(X = 0 \mid \lambda = 4.5) = 0.0111$

3 b Power $= \text{P}(\text{H}_0 \text{ is rejected} \mid \lambda = 4.1) = \text{P}(X = 0 \mid \lambda = 4.1)$

$$= \frac{e^{-4.1} 4.1^0}{0!} = 0.0166 \text{ (4 d.p.)}$$

4 $\text{H}_0 : \mu = 2 \quad \text{H}_1 : \mu \neq 2$

Assume H_0, so that $\bar{X} \sim \text{N}\left(2, \frac{0.004}{25}\right)$

Standardise the $\bar{X}$ variable

$$Z = \frac{\bar{X} - 2}{\sqrt{\frac{.004}{25}}} = \frac{5(\bar{X} - 2)}{0.06325}$$

Significance level 5%, so require 2.5% in each tail

From the tables, the critical region for Z is $Z > 1.96$ or $Z < -1.96$

So the critical values for $\bar{X}$ are given by

$$\frac{5(\bar{X} - 2)}{0.063245} = \pm 1.96$$

$\Rightarrow \bar{X} = 1.97521$ and $\bar{X} = 2.02479$

So the critical region for $\bar{X}$ is $\bar{X} < 1.97521$ or $\bar{X} > 2.02479$

Power $= \text{P}(\bar{X} < 1.97521 \mid \mu = 2.02) + 1 - \text{P}(\bar{X} < 2.02479 \mid \mu = 2.02)$

$= 0.0002 + 1 - 0.6475 = 0.3527$ (4 d.p.)

5 a $X \sim \text{B}(10, 0.4)$, critical region $X \geqslant 7$

Power $= \text{P}(X \geqslant 7 \mid p = 0.4) = 1 - \text{P}(X \leqslant 6 \mid p = 0.4)$

$= 1 - 0.9452 = 0.0548$

b $X \sim \text{B}(10, 0.8)$, critical region $X \geqslant 7$

Power $= \text{P}(X \geqslant 7 \mid p = 0.8)$

Let $Y \sim \text{B}(10, 0.2)$, then $\text{P}(Y \leqslant 3) = \text{P}(X \geqslant 7)$

So Power $= \text{P}(X \geqslant 7 \mid p = 0.8) = \text{P}(Y \leqslant 3) = 0.8791$

c The test is more powerful for values of p further away from $p = 0.3$.

6 a A Type I error is when H_0 is rejected when H_0 is in fact true.

b The size of a significance test is the probability of rejecting the null hypothesis when it is true:

Size = P(Type I error)

c Critical region is $X \geqslant 25$, $X \sim \text{N}(\mu, 10)$

Size $= \text{P}(\text{Type I error}) = \text{P}(X \geqslant 25 \mid \mu = 20)$

$= 1 - \text{P}(X < 25 \mid \mu = 20)$

$= 1 - 0.9431 = 0.0569$ (4 d.p.)

7 a $H_0 : p = 0.01 \quad H_1 : p > 0.01$

Assume H_0, so that $X \sim \text{Geo}(0.01)$

Significance level 5%

Require $P(X \leqslant c) < 0.05$

So $1-(1-0.01)^c < 0.05$

$(1-0.01)^c > 0.95$

$c \log 0.99 > \log 0.95$

$c < \dfrac{\log 0.95}{\log 0.99}$

$c < 5.10365$

So the critical value is 5 and the critical region is $X \leqslant 5$

b Power $= P(H_0 \text{ is rejected} \mid p = 0.2) = P(X \leqslant 5 \mid p = 0.2)$

$= 1-(1-0.2)^5 = 1-0.8^5 = 1-0.3277 = 0.6723$ (4 d.p.)

8 a $H_0 : p = 0.01 \quad H_1 : p \neq 0.01$

Assume H_0, so that $X \sim \text{Geo}(0.01)$

Significance level 5%

If $X = c_1$ is the lower boundary of the upper critical region, require $P(X \geqslant c_1) < 0.025$

So $(1-0.01)^{c_1-1} < 0.025$

$c_1 - 1 > \dfrac{\log 0.025}{\log 0.99}$

$c_1 > 368.04$

So $c_1 = 369$ and the upper critical region is $X \geqslant 369$

If c_2 is the upper boundary of the lower critical region, require $P(X \leqslant c_2) < 0.025$

So $1-(1-0.01)^{c_2} < 0.025$

$0.99^{c_2} > 0.975$

$c_2 < \dfrac{\log 0.975}{\log 0.99}$

$c_2 < 2.519$

So $c_2 = 2$ and the lower critical region is $X \leqslant 2$

So the critical region is $X \leqslant 2$ or $X \geqslant 369$

b Power $= P(H_0 \text{ is rejected} \mid p = 0.2) = P(X \leqslant 2 \mid p = 0.02) + P(X \geqslant 369 \mid p = 0.02)$

$= 1-(1-0.02)^2 + (1-0.02)^{368} = 1-0.98^2 + 0.98^{368}$

$= 1-0.9604 + 0.0006 = 0.0402$ (4 d.p.)

9 a Let the random variable X denote the number of defects found in a sample of 10 rolls, then $X \sim \text{Po}(8)$

Assume H_0, so that $X \sim \text{Po}(8)$

$$\text{Size} = \text{P(Type I error)} = \text{P}(X \geqslant 12 \mid X \sim \text{Po}(8)) + \text{P}(10 \leqslant X \leqslant 11 \mid X \sim \text{Po}(8)) \times \text{P}(X \geqslant 8 \mid X \sim \text{Po}(8))$$
$$= 1 - \text{P}(X \leqslant 11 \mid X \sim \text{Po}(8))$$
$$+ \left(\text{P}(X \leqslant 11 \mid X \sim \text{Po}(8)) - \text{P}(X \leqslant 9 \mid X \sim \text{Po}(8))\right) \times \left(1 - \text{P}(X \leqslant 7 \mid X \sim \text{Po}(8))\right)$$
$$= 1 - 0.8881 + ((0.8881 - 0.7166) \times (1 - 0.4530)$$
$$= 0.1119 + (0.1715 \times 0.547) = 0.1119 + 0.0938$$
$$= 0.2057$$

b $$\text{Power} = \text{P(H}_0 \text{ is rejected} \mid X \sim \text{Po}(10))$$
$$= \text{P}(X \geqslant 12 \mid X \sim \text{Po}(10)) + \text{P}(10 \leqslant X \leqslant 11 \mid X \sim \text{Po}(10)) \times \text{P}(X \geqslant 8 \mid X \sim \text{Po}(10))$$
$$= 1 - \text{P}(X \leqslant 11 \mid X \sim \text{Po}(10))$$
$$+ \left(\text{P}(X \leqslant 11 \mid X \sim \text{Po}(10)) - \text{P}(X \leqslant 9 \mid X \sim \text{Po}(10))\right) \times \left(1 - \text{P}(X \leqslant 7 \mid X \sim \text{Po}(10))\right)$$
$$= 1 - 0.6968 + ((0.6968 - 0.4579) \times (1 - 0.2202)$$
$$= 0.3032 + (0.2389 \times 0.7798) = 0.3032 + 0.1863$$
$$= 0.4895$$

10 a Let the random variable Y denote the number of jelly beans found in a box. Then the mean number of jelly beans in each box found in a sample of 20 boxes is given by

$\bar{Y} \sim \text{N}\left(\mu, \frac{5^2}{20}\right)$, where μ is the mean number of jelly beans in a box

The consumer group's null hypothesis is $\text{H}_0 = 80$ and it rejects the null hypothesis for $X \leqslant 10$ and $X \sim \text{Geo}(\text{P}(\bar{Y} < 79))$

Using a calculator $\text{P}(\bar{Y} < 79) = 0.185547\ldots$

$$\text{Size} = \text{P(Type I error)} = \text{P}(X \leqslant 10 \mid p = 0.185547\ldots)$$
$$= 1 - (1 - 0.185547)^{10} = 1 - 0.128428\ldots$$
$$= 0.8716 \text{ (4 d.p.)}$$

b For $\mu = 81$, then $\text{P}(\bar{Y} < 79) = 0.036819\ldots$

$$\text{Power} = \text{P(H}_0 \text{ is rejected} \mid p = 0.036819\ldots) = \text{P}(X \leqslant 10 \mid p = 0.036819\ldots)$$
$$= 1 - (1 - 0.036819)^{10} = 1 - 0.68719\ldots$$
$$= 0.3128 \text{ (4 d.p.)}$$

Challenge

a Model using a binomial distribution

$H_0 : p = 0.08 \quad H_1 : p > 0.08$

Assume H_0, so that $X \sim B(20, 0.08)$

Significance level 5%, so require c such that $P(X \geqslant c) < 0.05$

Using a calculator

$P(X \geqslant 4) = 1 - P(X \leqslant 3) = 1 - 0.9294 = 0.0706$

$P(X \geqslant 5) = 1 - P(X \leqslant 4) = 1 - 0.9817 = 0.0183$

$P(X \geqslant 4) > 0.05$ and $P(X \geqslant 5) < 0.05$, so critical region is $X \geqslant 5$

So the probability of failing any a test when the null hypothesis is true is 0.0183 and the probability of passing each test is 0.9817

$$\begin{aligned}\text{P(Type I error)} &= P(H_0 \text{ rejected} \mid H_0 \text{ true})\\ &= \text{P(fails test 1)} + \text{P(passes test 1 then fails test 2)} + \ldots\\ &\qquad + \text{P(passes first } n-1 \text{ tests then fails test } n)\\ &= 0.0183 + 0.9817 \times 0.0183 + 0.9817^2 \times 0.0183 + \ldots + 0.9817^{n-1} \times 0.0183\\ &= \sum_{i=0}^{n} 0.0183 \times 0.9817^{n-1}\end{aligned}$$

This is geometric series with first term 0.01834 and common ratio 0.98166, and using the formula for the sum of a finite geometric series gives

$$\text{P(Type I error)} = \frac{0.0183(1 - 0.9817^n)}{(1 - 0.9817)} = 1 - 0.9817^n$$

Require $\text{P(Type I error)} < 0.1$, so

$1 - 0.9817^n < 0.1 \Rightarrow 0.9817^n > 0.9$

$0.9817^6 = 0.8951\ldots$ and $0.9817^5 = 0.9118\ldots$

So maximum number of boxes that can be inspected is 5

b The probability of failing or passing the first two tests is 0.0183 and 0.9827 from part **a**.

The probability of failing test 3 or 4 when the actual distribution is $X \sim B(20, 0.2)$ is

$P(X \geqslant 5) = 1 - P(X \leqslant 4) = 1 - 0.6296 = 0.3704$, so the probability of passing test 3 or test 4 is 0.6296

$$\begin{aligned}\text{P(Type II error)} &= 1 - P(H_0 \text{ accepted} \mid H_0 \text{ false})\\ &= 1 - \text{P(passes first four tests)}\\ &= 1 - \left(\text{P(passes test 1)} \times \text{P(passes test 2)} \times \text{P(passes test 3)} \times \text{P(passes test 4)}\right)\\ &= 1 - (0.9817^2 0.6296^2)\\ &= 1 - 0.3820 = 0.6180\end{aligned}$$