Quality of tests 8A

1 a $H_0: p = 0.25 \quad H_1: p > 0.25$

Assume H_0, so that $X \sim B(10, 0.25)$

Significance level 5%, so require c such that $P(X \geqslant c) < 0.05$

From the binomial cumulative distribution tables

$P(X \geqslant 5) = 1 - P(X \leqslant 4) = 1 - 0.9219 = 0.0781$

$P(X \geqslant 6) = 1 - P(X \leqslant 5) = 1 - 0.9803 = 0.0197$

$P(X \geqslant 5) > 0.05$ and $P(X \geqslant 6) < 0.05$ so the critical value is 6

Hence the critical region is $X \geqslant 6$

b $P(\text{Type I error}) = P(X \geqslant 6 \mid p = 0.25) = 0.0197$

$P(\text{Type II error}) = P(X \leqslant 5 \mid p = 0.30) = 0.9527$

2 a $H_0: p = 0.30 \quad H_1: p < 0.30$

Assume H_0, so that $X \sim B(20, 0.30)$

Significance level 1%, so require c such that $P(X \leqslant c) < 0.01$

From the binomial cumulative distribution tables

$P(X \leqslant 2) = 0.0355$ and $P(X \leqslant 1) = 0.0076$

$P(X \leqslant 2) > 0.01$ and $P(X \leqslant 1) < 0.01$ so the critical value is 1

Hence the critical region is $X \leqslant 1$

b $P(\text{Type I error}) = P(X \leqslant 1 \mid p = 0.30) = 0.0076$

$P(\text{Type II error}) = P(X \geqslant 2 \mid p = 0.25) = 1 - P(X \leqslant 1 \mid p = 0.25)$

$= 1 - 0.0243 = 0.9757$

3 a $H_0: p = 0.45 \quad H_1: p \neq 0.45$

Assume H_0, so that $X \sim B(10, 0.45)$

Significance level 5%

If $X = c_1$ is the upper boundary of the lower critical region, require $P(X \leqslant c_1) < 0.025$

From the tables

$P(X \leqslant 1) = 0.0233$ and $P(X \leqslant 2) = 0.0996$

So $c_1 = 1$ and the lower critical region is $X \leqslant 1$

If $X = c_2$ is the lower boundary of the upper critical region, require $P(X \geqslant c_2) < 0.025$

From the tables

$P(X \geqslant 8) = 1 - P(X \leqslant 7) = 1 - 0.9726 = 0.0274$

$P(X \geqslant 9) = 1 - P(X \leqslant 8) = 1 - 0.9955 = 0.0045$

So $c_2 = 9$ and the upper critical region is $X \geqslant 9$

So the critical region is $X \leqslant 1$ or $X \geqslant 9$

3 b $\text{P(Type I error)} = \text{P}(X \leqslant 1 \mid p = 0.45) + \text{P}(X \geqslant 1 \mid p = 0.45)$
$= 0.0233 + 0.0045 = 0.0278$
$\text{P(Type II error)} = \text{P}(2 \leqslant X \leqslant 8 \mid p = 0.40) = \text{P}(X \leqslant 8 \mid p = 0.40) - \text{P}(X \leqslant 1 \mid p = 0.40)$
$= 0.9983 - 0.0464 = 0.9519$

4 a $\text{H}_0 : \lambda = 6 \quad \text{H}_1 : \lambda > 6$
Assume H_0, so that $X \sim \text{Po}(6)$
Significance level 5%, so require c such that $\text{P}(X \geqslant c) < 0.05$
From the Poisson cumulative distribution tables
$\text{P}(X \geqslant 10) = 1 - \text{P}(X \leqslant 9) = 1 - 0.9161 = 0.0839$
$\text{P}(X \geqslant 11) = 1 - \text{P}(X \leqslant 10) = 1 - 0.9574 = 0.0426$
$\text{P}(X \geqslant 10) > 0.05$ and $\text{P}(X \geqslant 11) < 0.05$ so the critical value is 11
Hence the critical region is $X \geqslant 11$

b $\text{P(Type I error)} = \text{P}(X \geqslant 11 \mid \lambda = 6) = 0.0426$
$\text{P(Type II error)} = \text{P}(X \leqslant 10 \mid \lambda = 7) = 0.9015$

5 a $\text{H}_0 : \lambda = 4.5 \quad \text{H}_1 : \lambda < 4.5$
Assume H_0, so that $X \sim \text{Po}(4.5)$
Significance level 5%, so require c such that $\text{P}(X \leqslant c) < 0.05$
From the Poisson cumulative distribution tables
$\text{P}(X \leqslant 1) = 0.0611$ and $\text{P}(X = 0) = 0.0111$
$\text{P}(X \leqslant 1) > 0.05$ and $\text{P}(X = 0) < 0.05$ so the critical value is 0
Hence the critical region is $X = 0$

b $\text{P(Type I error)} = \text{P}(X = 0 \mid \lambda = 4.5) = 0.0111$
$\text{P(Type II error)} = \text{P}(X \geqslant 0 \mid \lambda = 3.5) = 1 - \text{P}(X = 0 \mid \lambda = 3.5)$
$= 1 - 0.0302 = 0.9698$

6 a $\text{H}_0 : \lambda = 9 \quad \text{H}_1 : \lambda \neq 9$
Assume H_0, so that $X \sim \text{Po}(9)$
Significance level 5%
If $X = c_1$ is the upper boundary of the lower critical region, require $\text{P}(X \leqslant c_1) < 0.025$
From the tables $\text{P}(X \leqslant 4) = 0.0550$ and $\text{P}(X \leqslant 3) = 0.0212$
So $c_1 = 3$ and the lower critical region is $X \leqslant 3$

If $X = c_2$ is the lower boundary of the upper critical region, require $\text{P}(X \geqslant c_2) < 0.025$
$\text{P}(X \geqslant 15) = 1 - \text{P}(X \leqslant 14) = 1 - 0.9585 = 0.0415$
$\text{P}(X \geqslant 16) = 1 - \text{P}(X \leqslant 15) = 1 - 0.9780 = 0.0220$
So $c_2 = 16$ and the upper critical region is $X \geqslant 16$

So the critical region is $X \leqslant 3$ or $X \geqslant 16$

6 b $\text{P(Type I error)} = \text{P}(X \leqslant 3 \mid \lambda = 9) + \text{P}(X \geqslant 16 \mid \lambda = 9)$

$= 0.0212 + 0.0220 = 0.0432$

$\text{P(Type II error)} = \text{P}(4 \leqslant X \leqslant 15 \mid \lambda = 8) = \text{P}(X \leqslant 15 \mid \lambda = 8) - \text{P}(X \leqslant 3 \mid \lambda = 8)$

$= 0.9918 - 0.0424 = 0.9494$

7 a $\text{H}_0 : p = 0.2 \quad \text{H}_1 : p < 0.2$

Assume H_0, so that $X \sim \text{Geo}(0.2)$

Significance level 5%

Require $\text{P}(X \geqslant c) < 0.05$

So $(1-0.2)^{c-1} < 0.05$

$(c-1)\log 0.8 < \log 0.05$

$c - 1 > \dfrac{\log 0.05}{\log 0.8}$

$c > 14.425$

So the critical value is 15 and the critical region is $X \geqslant 15$

b $\text{P(Type I error)} = \text{P}(X \geqslant 15 \mid p = 0.2) = (1-0.2)^{15-1} = 0.8^{14} = 0.0440$ (4 d.p.)

$\text{P(Type II error)} = \text{P}(X \leqslant 14 \mid p = 0.05) = 1 - (1-0.05)^{14}$

$= 1 - 0.95^{14} = 1 - 0.4877 = 0.5123$ (4 d.p.)

8 a $\text{H}_0 : p = 0.02 \quad \text{H}_1 : p < 0.02$

Assume H_0, so that $X \sim \text{Geo}(0.02)$

Significance level 1%

Require $\text{P}(X \geqslant c) < 0.01$

So $(1-0.02)^{c-1} < 0.01$

$(c-1)\log 0.98 < \log 0.01$

$c - 1 > \dfrac{\log 0.01}{\log 0.98}$

$c > 228.948$

So the critical value is 229 and the critical region is $X \geqslant 229$

b $\text{P(Type I error)} = \text{P}(X \geqslant 229 \mid p = 0.02) = (1-0.02)^{229-1} = 0.98^{228} = 0.0100$ (4 d.p.)

$\text{P(Type II error)} = \text{P}(X \leqslant 228 \mid p = 0.01) = 1 - (1-0.01)^{228}$

$= 1 - 0.99^{228} = 1 - 0.1011 = 0.8989$ (4 d.p.)

9 a $H_0 : p = 0.01 \quad H_1 : p \neq 0.01$

Assume H_0, so that $X \sim \text{Geo}(0.01)$

Significance level 5%

If $X = c_1$ is the lower boundary of the upper critical region, require $P(X \geqslant c_1) < 0.025$

So $(1-0.01)^{c_1-1} < 0.025$

$c_1 - 1 > \dfrac{\log 0.025}{\log 0.99}$

$c_1 > 368.04$

So $c_1 = 369$ and the upper critical region is $X \geqslant 369$

If c_2 is the upper boundary of the lower critical region, require $P(X \leqslant c_2) < 0.025$

So $1-(1-0.01)^{c_2} < 0.025$

$0.99^{c_2} > 0.975$

$c_2 < \dfrac{\log 0.975}{\log 0.99}$

$c_2 < 2.519$

So $c_2 = 2$ and the lower critical region is $X \leqslant 2$

So the critical region is $X \leqslant 2$ or $X \geqslant 369$

b $\text{P(Type I error)} = P(X \leqslant 2 \mid p = 0.01) + P(X \geqslant 369 \mid p = 0.01)$

$= 1-(1-0.01)^2 + (1-0.01)^{369-1} = 1-0.99^2 + 0.99^{368}$

$= 1-0.9801+0.0248 = 0.0447$ (4 d.p.)

$\text{P(Type II error)} = P(3 \leqslant X \leqslant 368 \mid p = 0.1)$

$= P(X \leqslant 368 \mid p = 0.1) - P(X \leqslant 2 \mid p = 0.1)$

$= 1-0.9^{368} - \left(1-0.9^2\right)$

$= 0.9^2 - 0.9^{368} = 0.8100$ (4 d.p.)

10 a i A Type 1 error occurs when H_0 is rejected but H_0 is in fact true.

ii A Type 2 error occurs when H_0 is accepted but H_0 is in fact false.

10 b $H_0 : p = 0.004 \quad H_1 : p \neq 0.004$

Assume H_0, so that $X \sim \text{Geo}(0.009)$

Significance level 10%

If $X = c_1$ is the lower boundary of the upper critical region, require $P(X \geqslant c_1)$ to be as close as possible to 5%. First, find $P(X \geqslant a) < 0.05$

So $(1-0.004)^{a-1} < 0.05$

$$a - 1 > \frac{\log 0.05}{\log 0.996} \Rightarrow a > 748.434$$

So c_1 could be 748 or 749

$P(X \geqslant 748) = 0.996^{747} = 0.05009$ and $P(X \geqslant 749) = 0.996^{748} = 0.04987$,

and as $P(X \geqslant 748)$ is closer to 0.05, the upper critical region is $X \geqslant 748$

If c_2 is the upper boundary of the lower critical region, require $P(X \leqslant c_2)$ to be as close as possible to 5%. First, find $P(X \leqslant b) < 0.05$

So $1-(1-0.004)^b < 0.05$

$0.996^b > 0.95$

$$b < \frac{\log 0.95}{\log 0.996} \Rightarrow b < 12.79$$

So c_2 could be 12 or 13; $P(X \leqslant 12) = 1 - 0.996^{12} = 0.04696$ and $P(X \leqslant 13) = 1 - 0.996^{13} = 0.05077$

and as $P(X \leqslant 13)$ is closer to 0.05, the lower critical region is $X \leqslant 13$

So the critical region is $X \leqslant 13$ or $X \geqslant 748$

c $P(\text{Type I error}) = P(X \leqslant 13 \mid p = 0.004) + P(X \geqslant 748 \mid p = 0.004)$

$= 0.05077 + 0.05009 = 0.1009$ (4 d.p.)

11 a $H_0 : p = 0.05 \quad H_1 : p > 0.05$

Assume H_0, so that $X \sim B(40, 0.05)$

Significance level 5%, so require c such that $P(X \geq c) < 0.05$

From the binomial cumulative distribution tables

$P(X \geqslant 4) = 1 - P(X \leqslant 3) = 1 - 0.8619 = 0.1381$

$P(X \geqslant 5) = 1 - P(X \leqslant 4) = 1 - 0.9520 = 0.0480$

$P(X \geqslant 4) > 0.05$ and $P(X \geqslant 5) < 0.05$ so the critical value is 5

Hence the critical region is $X \geqslant 5$

b $P(\text{Type I error}) = P(X \geqslant 5 \mid p = 0.05) = 0.0480$

11 c $\text{H}_0 : p = 0.05 \quad \text{H}_1 : p > 0.05$

Assume H_0, so that $X \sim \text{Geo}(0.05)$

Significance level 5%

Require $\text{P}(X \geqslant c) \leqslant 0.05$

So $1-(1-0.05)^c \leqslant 0.05$

$0.95^c \geqslant 0.95$

$c \log 0.95 \geqslant \log 0.95$

$c \leqslant \dfrac{\log 0.95}{\log 0.95}$

$c = 1$

So the critical region is $X = 2$

d $\text{P(Type I error)} = \text{P}(X = 1 \mid p = 0.05) = 0.05$

e For David's test $X \sim \text{Geo}(0.0588)$

$\text{P(Type II error)} = \text{P}(X > 1 \mid p = 0.0588) = (1-0.0588)^1 = 0.9412$

f For Michael's test $X \sim \text{B}(40, 0.0588)$

Find the probability by using a statistical calculator

$\text{P(Type II error)} = \text{P}(X \leqslant 4 \mid p = 0.0588) = 0.9162$ (4 d.p.)