Probability generating functions Mixed exercise 7

1 a Coefficients must sum to 1, i.e. $G_Y(1) = 1$

Hence, $k(2+2+3) = 1 \Rightarrow k = \frac{1}{7}$

b $P(Y=1)$ is the coefficient of t in the probability generating function h

Hence, $P(Y=1) = \frac{2}{7}$

c As X and Y are independent, $G_Z(t) = G_X(t)G_Y(t)$

$$G_Z(t) = \frac{1}{7}\left(2+2t+3t^2\right)\times\frac{1}{16}(1+t+2t^2)^2 = \frac{1}{112}\left(2+2t+3t^2\right)\left(1+t+2t^2\right)^2$$

d $P(Z=2)$ is the coefficient of t^2 in $G_Z(t)$, expanding $G_Z(t)$ gives

$$G_Z(t) = \frac{1}{112}\left(2+2t+3t^2\right)\left(1+t+2t^2\right)\left(1+t+2t^2\right)$$
$$= \frac{1}{112}\left(2+2t+3t^2\right)\left(1+2t+5t^2+4t^3+4t^4\right)$$

So the coefficient of t^2 in $G_Z(t)$ is $\frac{1}{112}(4+3+10) = \frac{17}{112} = 0.1518$ (4 d.p.)

2 The standard formula for the probability generating function of a geometric distribution is:

$$G_X(t) = \frac{pt}{1-(1-p)t}$$

Using the quotient rule

$$G'_X(t) = \frac{(1-(1-p)t)p + pt(1-p)}{(1-(1-p)t)^2} = \frac{p - pt + p^2t + pt - p^2t}{(1-(1-p)t)^2} = \frac{p}{(1-(1-p)t)^2}$$

$$E(X) = G'_X(1) = \frac{p}{(1-(1-p))^2} = \frac{1}{p}$$

$$G''_X(t) = \frac{2p(1-p)}{(1-(1-p)t)^3}$$

$$G''_X(1) = \frac{2(1-p)}{p^2}$$

$$\text{Var}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2$$
$$= \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2-2p+p-1}{p^2}$$
$$= \frac{1-p}{p^2}$$

3 If $X \sim \mathrm{B}(5, 0.4)$, then $\mathrm{P}(X = x) = \binom{5}{x} 0.4^x (1-0.4)^{5-x} = \binom{5}{x} 0.4^x (0.6)^{5-x}$

So $\mathrm{G}_X(t) = \sum_{x=0}^{5} \binom{5}{x} 0.4^x (0.6)^{n-x} t^x = \sum_{x=1}^{5} \binom{5}{x} (0.4t)^x (0.6)^{n-x}$

$$= (0.6)^5 + \binom{5}{1}(0.6)^4(0.4t)^1 + \binom{5}{2}(0.6)^3(0.4t)^2 + \binom{5}{3}(0.6)^2(0.4t)^3 + \binom{5}{4}(0.6)(0.4t)^4 + (0.4t)^5$$

This is a binomial expansion of the form $(a+b)^n$, with $n = 5, a = 0.6$ and $b = 0.4t$

So $\mathrm{G}_X(t) = (0.6 + 0.4t)^5$

An alternative approach is to let the random variable Y be the outcome of one trial. Then P(Y = 1) =0.4 and P(Y = 0) = 0.6, and the probability generating function for Y is:
$\mathrm{G}_Y(t) = 0.6 + 0.4t$ Now note that as each trial is independent X is equal to the sum of 5 independent trials, so $\mathrm{G}_X(t) = (\mathrm{G}_Y(t))^5 = (0.6 + 0.4t)^5$

4 a X has a geometric distribution, $X \sim \mathrm{Geo}\left(\frac{4}{15}\right)$

b Using the standard formula for the probability generating function of a geometric distribution:

$$\mathrm{G}_X(t) = \frac{\frac{4}{15}t}{1 - \left(1 - \frac{4}{15}\right)t} = \frac{4t}{15 - 11t}$$

c i Using the product rule

$$\mathrm{G}'_X(t) = \frac{4}{15-11t} + \frac{4t}{(15-11t)^2}(-11)(-1)$$
$$= \frac{4(15 - 11t + 11t)}{(15-11t)^2} = \frac{60}{(15-11t)^2}$$
$$\mathrm{E}(X) = \mathrm{G}'_X(1) = \frac{60}{4^2} = \frac{15}{4} = 3.75$$

ii $\mathrm{G}''_X(t) = 60(15-11t)^{-3}(-2)(-11) = 1320(15-11t)^{-3}$

$$\mathrm{G}''_X(1) = \frac{1320}{4^3} = \frac{165}{8}$$
$$\mathrm{Var}(X) = \mathrm{G}''_X(1) + \mathrm{G}'_X(1) - (\mathrm{G}'_X(1))^2$$
$$= \frac{165}{8} + \frac{15}{4} - \frac{225}{16} = \frac{330 + 60 - 225}{16}$$
$$= \frac{165}{16} = 10.3125$$

4 d Let the random variable Y denote the number of selections from the second box until Fluffy finds a fishy treat, so $Y \sim \text{Geo}\left(\frac{5}{12}\right)$

$$G_Y(t) = \frac{\frac{5}{12}t}{1-\frac{7}{12}t} = \frac{5t}{12-7t}$$

As X and Y are independent, $Z = X + Y$ and $G_Z(t) = G_X(t) \times G_Y(t)$, so

$$G_Z(t) = \frac{4t}{15-11t} \times \frac{5t}{12-7t} = \frac{20t^2}{(15-11t)(12-7t)}$$

e $$G'_Z(t) = \frac{20}{(15-11t)(12-7t)}\left[2t + t^2\left(\frac{(-11)(-1)}{15-11t} + \frac{(-7)(-1)}{12-7t}\right)\right]$$

$$= \frac{20t}{(15-11t)^2(12-7t)^2}[2(15-11t)(12-7t) + 11t(12-7t) + 7t(15-11t)]$$

$$= \frac{20t(360-237t)}{(15-11t)^2(12-7t)^2}$$

$$E(Z) = G'_Z(1) = \frac{20 \times 123}{4^2 \times 5^2} = \frac{123}{20} = 6.15$$

$$G''_Z(t) = \frac{20}{(15-11t)^2(12-7t)^2}\left(360 - 474t + t(360-237t)\left(\frac{(-11)(-2)}{15-11t} + \frac{(-7)(-2)}{12-7t}\right)\right)$$

$$G''_Z(1) = \frac{20}{4^2 \times 5^2}\left(360 - 474 + 123\left(\frac{22}{4} + \frac{14}{5}\right)\right) = \frac{1}{20}\left(\frac{123(110+56)}{20} - \frac{114 \times 20}{20}\right)$$

$$= \frac{1}{400}(20418 - 2280) = \frac{18138}{400} = \frac{9069}{200}$$

$$\text{Var}(Z) = G''_Z(1) + G'_Z(1) - (G'_Z(1))^2 = \frac{9069}{200} + \frac{123}{20} - \left(\frac{123}{20}\right)^2$$

$$= \frac{18138 + 2460 - 15129}{400} = \frac{5469}{400}$$

Standard deviation of $Z = \sqrt{\text{Var}(Z)} = \frac{\sqrt{5469}}{20} = 3.6976$ (4 d.p.)

5 a i $X \sim \text{Po}(0.5)$ so $P(X = 0) = \frac{e^{-0.5}0.5^0}{0!} = e^{-0.5}$

ii $P(X \geqslant 3) = 1 - P(X \leqslant 2) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$

so $P(X \geqslant 3) = 1 - e^{-0.5} - \frac{e^{-0.5}0.5^1}{1} - \frac{e^{-0.5}0.5^2}{2} = 1 - e^{-0.5}\left(1 + \frac{1}{2} + \frac{1}{8}\right) = 1 - \frac{13}{8}e^{-0.5}$

5 b $P(Y=0)=P(X=0)=e^{-0.5}$

$P(Y=1)=P(X=1)=\left(e^{-0.5}\times 0.5\right)=\frac{1}{2}e^{-0.5}$

$P(Y=2)=P(X=2)=\left(e^{-0.5}\times\frac{0.5^2}{2}\right)=\frac{1}{8}e^{-0.5}$

$P(Y=3)=P(X\geqslant 3)=1-\frac{13}{8}e^{-0.5}$

Hence $G_Y(t)=e^{-0.5}+\frac{e^{-0.5}}{2}t+\frac{e^{-0.5}}{8}t^2+\left(1-\frac{13e^{-0.5}}{8}\right)t^3$

$=t^3+e^{-0.5}\left(1+\frac{1}{2}t+\frac{1}{8}t^2-\frac{13}{8}t^3\right)$

c $G'_Y(t)=3t^2+e^{-0.5}\left(\frac{1}{2}+\frac{1}{4}t-\frac{39}{8}t^2\right)$

$E(Y)=G'_Y(1)=3t^2+e^{-0.5}\left(\frac{1}{2}+\frac{1}{4}-\frac{39}{8}\right)=3-\frac{33}{8}e^{-0.5}=0.4981\ldots=0.498\text{ (3 d.p.)}$

$G''_Y(t)=6t+e^{-0.5}\left(\frac{1}{4}-\frac{39}{4}t\right)$

So $G''_Y(1)=6-\frac{19}{2}e^{-0.5}=0.2380\text{ (4 d.p.)}$

$\text{Var}(Y)=G''_Y(1)+G'_Y(1)-(G'_Y(1))^2=0.2380+0.4981-(0.4981)^2=0.488\text{ (3 d.p.)}$

6 a Using the product rule,

$G'_X(t)=\frac{5t^2}{(2-t)^6}+\frac{2t}{(2-t)^5}=\frac{5t^2+2t(2-t)}{(2-t)^6}=\frac{3t^2+4t}{(2-t)^6}$

So $E(X)=G'_X(1)=3+4=7$

$G''_X(t)=\frac{6(3t^2+4t)}{(2-t)^7}+\frac{6t+4}{(2-t)^6}=\frac{18t^2+24t+(2-t)(6t+4)}{(2-t)^7}$

$=\frac{12t^2+32t+8}{(2-t)^7}$

$G''_X(1)=12+32+8=52$

$\text{Var}(X)=G''_X(1)+G'_X(1)-(G'_X(1))^2=52+7-7^2=10$

Standard deviation of $X=\sqrt{\text{Var}(X)}=\sqrt{10}\frac{\sqrt{5469}}{20}=3.1623\text{ (4 d.p.)}$

b i $P(X=0)=G_X(0)$

So $P(X=0)=G_X(0)=\frac{0}{2^5}=0$

6 b ii $\mathrm{P}(X=2)=\frac{1}{2!}\mathrm{G}''_X(0)$

So $\mathrm{P}(X=2)=\frac{1}{2}\mathrm{G}''_X(0)=\frac{1}{2}\times\frac{8}{2^7}=\frac{2^3}{2^8}=\frac{1}{2^5}=\frac{1}{32}$

c i If $Y=aX+b$, where a and b are positive integers, then $\mathrm{G}_Y(t)=t^b\mathrm{G}_X(t^a)$

So $\mathrm{G}_{2Y-1}(t)=t^{-1}\mathrm{G}_Y\left(t^2\right)=\frac{1}{t}\times\frac{t^2}{\left(4-3t^2\right)^2}=\frac{t}{\left(4-3t^2\right)^2}$

ii As X and Y are independent random variables, $\mathrm{G}_Z(t)=\mathrm{G}_X(t)\times\mathrm{G}_Y(t)$ Hence

$\mathrm{G}_Z(t)=\frac{t^2}{(2-t)^5}\times\frac{t}{(4-3t)^2}=\frac{t^3}{(2-t)^5(4-3t)^2}$

d Rewrite $\mathrm{G}_Z(t)$ as $\mathrm{G}_Z(t)=t^3(2-t)^{-5}(4-3t)^{-2}$ and use the product rule to compute:

$\mathrm{G}'_Z(t)=3t^2(2-t)^{-5}(4-3t)^{-2}+(-5)(-1)t^3(2-t)^{-6}(4-3t)^{-2}+(-2)(-3)t^3(2-t)^{-5}(4-3t)^{-3}$

$=\frac{3t^2}{(2-t)^5(4-3t)^2}+\frac{5t^3}{(2-t)^6(4-3t)^2}+\frac{6t^3}{(2-t)^5(4-3t)^3}$

So $\mathrm{E}(Z)=\mathrm{G}'_Z(1)=3+5+6=14$

7 a $\mathrm{G}_X(1)=1$, so $k(1+2+3)^2=1$

This gives $36k=1\Rightarrow k=\frac{1}{36}$

b $\mathrm{G}_X(t)=k\left(1+2t^2+3t^3\right)\left(1+2t^2+3t^3\right)=k\left(1+4t^2+6t^3+4t^4+12t^5+9t^6\right)$

Coefficient of t^4 is $4k=\frac{1}{9}$, so $\mathrm{P}(X=4)=\frac{1}{9}$

c $\mathrm{G}'_X(t)=\frac{1}{36}(8t+18t^2+16t^3+60t^4+54t^5)$

$\mathrm{E}(X)=\mathrm{G}'_X(1)=\frac{1}{36}(8+18+16+60+54)=\frac{156}{36}=\frac{52}{12}=\frac{13}{3}$

$\mathrm{G}''_X(t)=\frac{1}{36}\left(8+36t+48t^2+240t^3+270t^4\right)$

$\mathrm{G}''_X(1)=\frac{1}{36}(8+36+48+240+270)=\frac{602}{36}=\frac{301}{18}$

$\mathrm{Var}(X)=\mathrm{G}''_X(1)+\mathrm{G}'_X(1)-(\mathrm{G}'_X(1))^2=\frac{301}{18}+\frac{13}{3}-\frac{169}{9}$

$=\frac{301+78-338}{18}=\frac{41}{18}=2.2778$ (4 d.p.)

d $\mathrm{G}_{3X-2}(t)=t^{-2}\mathrm{G}_X\left(t^3\right)=\frac{1}{36}t^{-2}\left(1+2t^6+3t^9\right)^2$

Challenge

a Use the general result $\mathrm{P}(X=x)=\frac{1}{n!}G_X^{(n)}(0)$

i $\mathrm{P}(X=0)=\mathrm{G}_X(0)$

So $\mathrm{P}(X=0)=\mathrm{G}_X(0)=\tan(0)=0$

ii $\mathrm{P}(X=1)=\mathrm{G}'_X(0)$

Using the chain rule,

$$\mathrm{G}'_X(t)=\frac{\pi}{4}\sec^2\left(\frac{\pi t}{4}\right)$$

So $\mathrm{P}(X=1)=\mathrm{G}'_X(0)=\frac{\pi}{4}\sec^2(0)=\frac{\pi}{4}$

iii $\mathrm{P}(X=2)=\frac{1}{2!}\mathrm{G}''_X(0)$

Using the chain rule,

$$\mathrm{G}''_X(t)=\left(\frac{\pi}{4}\right)^2 2\sec^2\left(\frac{\pi t}{4}\right)\tan\left(\frac{\pi t}{4}\right)$$

So $\mathrm{P}(X=2)=\frac{1}{2}\mathrm{G}''_X(0)=\left(\frac{\pi}{4}\right)^2\sec^2(0)\tan(0)=0$

b $\mathrm{E}(X)=\mathrm{G}'_X(1)=\frac{\pi}{4}\sec^2\left(\frac{\pi}{4}\right)=\frac{2\pi}{4}=\frac{\pi}{2}$

$$\mathrm{G}''_X(1)=2\left(\frac{\pi}{4}\right)^2\sec^2\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)=2\left(\frac{\pi}{4}\right)^2 2=4\left(\frac{\pi}{4}\right)^2=\frac{\pi^2}{4}$$

$$\mathrm{Var}(X)=\mathrm{G}''_X(1)+\mathrm{G}'_X(1)-(\mathrm{G}'_X(1))^2=\frac{\pi^2}{4}+\frac{\pi}{2}-\left(\frac{\pi}{2}\right)^2=\frac{\pi^2}{4}+\frac{\pi}{2}-\frac{\pi^2}{4}=\frac{\pi}{2}$$

c $\mathrm{P}(X=3)=\frac{1}{3!}\mathrm{G}'''_X(0)$

$$\mathrm{G}'''_X(t)=2\left(\frac{\pi}{4}\right)^3\sec^2\left(\frac{\pi t}{4}\right)\sec^2\left(\frac{\pi t}{4}\right)+4\left(\frac{\pi}{4}\right)^3\sec^2\left(\frac{\pi t}{4}\right)\tan\left(\frac{\pi t}{4}\right)\tan\left(\frac{\pi t}{4}\right)$$

$$=2\left(\frac{\pi}{4}\right)^3\sec^4\left(\frac{\pi t}{4}\right)+4\left(\frac{\pi}{4}\right)^3\sec^2\left(\frac{\pi t}{4}\right)\tan^2\left(\frac{\pi t}{4}\right)$$

So $\mathrm{P}(X=3)=\frac{1}{6}\mathrm{G}''_X(0)=\frac{2}{6}\left(\frac{\pi}{4}\right)^3\sec^4(0)+\frac{4}{6}\left(\frac{\pi}{4}\right)^3\sec^2(0)\tan^2(0)=\frac{\pi^3}{3\times 64}=\frac{\pi^3}{192}$