Probability generating functions 7D

1 a $Z = X + Y,\quad G_Z(t) = G_X(t)G_Y(t)$

$$G_Z(t) = \left(\frac{1}{4}t + \frac{1}{4}t^2 + \frac{1}{2}t^3\right)\left(\frac{5}{6} + \frac{1}{6}t\right)$$
$$= \frac{t}{24}\left(1 + t + 2t^2\right)(5 + t)$$
$$= \frac{t}{24}\left(5 + 6t + 11t^2 + 2t^3\right)$$
$$= \frac{5}{24}t + \frac{1}{4}t^2 + \frac{11}{24}t^3 + \frac{1}{12}t^4$$

b $G_Z'(t) = \frac{1}{24}\left(5 + 12t + 33t^2 + 8t^3\right)$

$$E(Z) = G_Z'(1) = \frac{1}{24}(5 + 12 + 33 + 8) = \frac{58}{24} = \frac{29}{12}$$
$$G_X'(t) = \frac{1}{4} + \frac{1}{2}t + \frac{3}{2}t^2 \Rightarrow E(X) = G_X'(1) = \frac{1}{4} + \frac{1}{2} + \frac{3}{2} = \frac{9}{4}$$
$$G_Y'(t) = \frac{1}{6} \Rightarrow E(Y) = G_Y'(1) = \frac{1}{6}$$
$$E(X) + E(Y) = \frac{9}{4} + \frac{1}{6} = \frac{54 + 4}{24} = \frac{58}{24} = \frac{29}{12} = E(Z)$$

2 a $G_X(t) = \frac{1}{4}(1 + t)^2 = (0.5)^2(1 + t)^2 = (0.5 + 0.5t)^2 = (1 - 0.5 + 0.5t)^2$

This the standard probability generating function for the binomial distribution, with $p = 0.5$ and $n = 2$, so $X \sim B(2, 0.5)$

$$G_Y(t) = \frac{1}{25}(1 + t)^2 = (0.2)^2(2 + 3)^2 = (0.4 + 0.6t)^2 = (1 - 0.6 + 0.6t)^2$$

This the standard probability generating function for the binomial distribution, with $p = 0.6$ and $n = 2$, so $X \sim B(2, 0.6)$

b $Z = X + Y,\quad G_Z(t) = G_X(t)G_Y(t)$

$$G_Z(t) = G_X(t)G_Y(t)$$
$$= \frac{1}{4}(1 + t)^2 \times \frac{1}{25}(2 + 3t)^2$$
$$= \frac{1}{100}(1 + 2t + t^2) \times (4 + 12t + 9t^2)$$
$$= \frac{1}{100}(4 + 20t + 37t^2 + 30t^3 + 9t^4)$$
$$= \frac{1}{25} + \frac{1}{5}t + \frac{37}{100}t^2 + \frac{3}{10}t^3 + \frac{9}{100}t^4$$

2 c $G'_Z(t)=\frac{1}{100}\left(20+74t+90t^2+36t^3\right)$

$E(Z)=G'_Z(1)=\frac{1}{100}(20+74+90+36)=\frac{220}{100}=\frac{11}{5}$

$G'_X(t)=\frac{1}{2}(1+t),\ E(X)=G'_X(1)=1$

$G'_Y(t)=\frac{6}{25}(2+3t),\ E(Y)=G'_Y(1)=\frac{30}{25}=\frac{6}{5}$

$E(Y)+E(X)=1+\frac{6}{5}=\frac{11}{5}=E(Z)$

3 a Using the standard formula for the probability generating function of a Poisson distribution,
$G_X(t)=e^{1.3(t-1)}$ and $G_Y(t)=e^{2.4(t-1)}$

b By independence, $G_Z(t)=G_X(t)G_Y(t)$
So $G_Z(t)=e^{1.3(t-1)}e^{2.4(t-1)}\Rightarrow G_Z(t)=e^{3.7(t-1)}$

c $G'_Z(t)=3.7e^{3.7(t-1)}$
$E(Z)=G'_Z(1)=3.7$
Similarly, $G'_X(t)=1.3e^{1.3(t-1)}$ and $G'_Y(t)=2.4e^{2.4(t-1)}$
So $E(X)+E(Y)=G'_X(1)+G'_Y(1)=1.3+2.4=3.7=E(Z)$

4 a Let the random variable X denote the number of rolls of a fair six-sided dice until a five is thrown, so $X\sim\text{Geo}\left(\frac{1}{6}\right)$

From the properties of a geometric distribution $P(X=x)=\frac{1}{6}\left(\frac{5}{6}\right)^{x-1}$

$$G_X(t)=\sum P(X=x)t^x=\sum_{x=1}^{\infty}\frac{1}{6}\left(\frac{5}{6}\right)^{x-1}t^x$$

$$=\frac{t}{6}\sum_{x=1}^{\infty}\left(\frac{5t}{6}\right)^{x-1}=\frac{t}{6}\sum_{x=0}^{\infty}\left(\frac{5t}{6}\right)^{x}\qquad\text{(the sum is a geometric series)}$$

$$=\frac{t}{6}\frac{1}{1-\frac{5}{6}t}=\frac{t}{6-5t}\qquad\text{(using the formula for the sum of a convergent geometric series)}$$

b Let the random variable Y denote the number of rolls of a fair ten-sided dice until two fives have been thrown, so $Y\sim\text{Negative B}(2,0.1)$
Using the standard formula for the probability generating function of a negative binomial

$$G_Y(t)=\left(\frac{0.1t}{1-(1-0.1)t}\right)^2=\left(\frac{0.1t}{1-0.9t}\right)^2=\left(\frac{t}{10-9t}\right)^2$$

c $Z=X+Y,\ G_Z(t)=G_X(t)G_Y(t)$

$$G_Z(t)=\frac{t}{6-5t}\left(\frac{t}{10-9t}\right)^2=\frac{t^3}{(6-5t)(10-9t)^2}$$

4 d Rewrite $G_Z(t) = t^3(6-5t)^{-1}(10-9t)^{-2}$ and differentiate using the product rule,

$$G'_Z(t) = \frac{3t^2}{(6-5t)(10-9t)^2} + \frac{(-1)(-5)t^3}{(6-5t)^2(10-9t)^2} + \frac{(-2)(-9)t^3}{(6-5t)(10-9t)^3}$$

$$G'_Z(1) = 3 + (-1)(-5) + (-2)(-9) = 3 + 5 + 18 = 26$$

Alternatively, find $G'_X(1)$ and $G'_Y(1)$ and add the results using the fact that $\mathrm{E}(Z) = \mathrm{E}(X) + \mathrm{E}(Y)$

$G_X(t) = t(6-5t)^{-1}$ $G'_X(t) = (6-5t)^{-1} + (-1)(-5)t\,(6-5t)^{-2}$ $G'_X(1) = 1 + 5 = 6$

$G_Y(t) = t^2(10-9t)^{-2}$ $G'_Y(t) = 2t(10-9t)^{-2} + (-2)(-9)t\,(10-9t)^{-3}$ $G'_Y(1) = 2 + 18 = 20$

$\mathrm{E}(X) + \mathrm{E}(Y) = G'_X(1) + G'_Y(1) = 6 + 20 = 26 = \mathrm{E}(Z)$

5 a $G_X(t) = k(1+2t)^3$

$G_X(1) = 1$, so $k(1+2)^3 = 1$

That is $27k = 1$, hence $k = \frac{1}{27}$

b $\mathrm{P}(X = 2)$ is the coefficient of t^2 in $G_X(t)$, expanding $G_X(t)$ gives

$G_X(t) = k(1+2t)^3 = k(1 + 6t + 3(2t)^2 + (2t)^3) = k\left(1 + 6t + 12t^2 + 8t^3\right)$

So $\mathrm{P}(X = 2) = 12k = \frac{12}{27} = \frac{4}{9}$

c Using chain rule gives $G'_X(t) = 6k(1+2t)^2$

So $\mathrm{E}(X) = G'_X(1) = \frac{6}{27} \times 3^2 = \frac{54}{27} = 2$

$G''_X(t) = 24k(1+2t)$, so $G''_X(1) = \frac{24 \times 3}{27} = \frac{8}{3}$

Hence, $\mathrm{Var}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 = \frac{8}{3} + 2 - 4 = \frac{2}{3}$

d $\mathrm{E}(Y) = G'_Y(1)$ and $G'_Y(t) = \frac{3}{4}$ hence $\mathrm{E}(Y) = \frac{3}{4}$

$\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y) = 2 + \frac{3}{4} = \frac{11}{4}$

6 a $G_Z(t) = G_X(t)G_Y(t) = \dfrac{4t}{(3-t)^2(3-2t)^3}$

b $G_X(t) = 4(3-t)^{-2}$ $G'_X(t) = 8(3-t)^{-3}$ $G'_X(1) = 8(2)^{-3} = 1$

$G_Y(t) = t(3-2t)^{-3}$ $G'_Y(t) = (3-2t)^{-3} + 6t(3-2t)^{-4}$ $G'_Y(1) = 1 + 6 = 7$

$\mathrm{E}(Z) = \mathrm{E}(X) + \mathrm{E}(Y) = G'_X(1) + G'_Y(1) = 1 + 7 = 8$

6 c $G_Z(t) = \dfrac{4t}{(3-t)^2(3-2t)^3}$

$G'_Z(t) = \dfrac{4}{(3-t)^2(3-2t)^3} + \dfrac{4t}{(3-t)^2}(3-2t)^{-4}(-3)(-2) + \dfrac{4t}{(3-2t)^3}(3-t)^{-3}(-2)(-1)$

$= \dfrac{4}{(3-t)^3(3-2t)^4}\left((3-t)(3-2t) + 6t(3-t) + 2t(3-2t)\right)$

$= 4 \times \dfrac{9+15t-8t^2}{(3-t)^3(3-2t)^4}$

$G''_Z(t) = \dfrac{4(15-16t)}{(3-t)^3(3-2t)^4} + 4(-4)(-2) \times \dfrac{9+15t-8t^2}{(3-t)^3(3-2t)^5} + 4(-3)(-1) \times \dfrac{9+15t-8t^2}{(3-t)^4(3-2t)^4}$

$G''_Z(1) = \dfrac{4(-1)}{2^3} + \dfrac{32 \times 16}{2^3} + \dfrac{12 \times 16}{2^4} = -\dfrac{1}{2} + 64 + 12 = \dfrac{151}{2}$

$\text{Var}(Z) = G''_Z(1) + G'_Z(1) - (G'_Z(1))^2$

$= \dfrac{151}{2} + 8 - 64 = \dfrac{151-112}{2} = \dfrac{39}{2}$

7 a Let the random variables Y and Z denote the number of prizes won by Adrian and Chloe respectively from their 5 scratch cards, so $Y \sim \text{B}(5, 0.3)$ and $Z \sim \text{B}(5, 0.4)$

$G_Y(t) = (1 - 0.3 + 0.3t)^5 = (0.7 + 0.3t)^5$

$G_Z(t) = (1 - 0.4 + 0.4t)^5 = (0.6 + 0.4t)^5$

As the games are independent

$G_X(t) = G_Y(t)G_Z(t) = (0.7 + 0.3t)^5(0.6 + 0.4t)^5$

b $G'_X(t) = 5(0.3)(0.7+0.3t)^4(0.6+0.4t)^5 + 5(0.4)(0.7+0.3t)^5(0.6+0.4t)^4$

$\text{E}(X) = G'_X(1) = 5(0.3) + 5(0.4) = 1.5 + 2.0 = 3.5$

8 a $G_Y(t) = G_X(t^3) = \dfrac{1}{2}t^3 + \dfrac{1}{2}\left(t^3\right)^3 = \dfrac{1}{2}t^3 + \dfrac{1}{2}t^9$

b $G_Y(t) = t^3 G_X\left(t^2\right) = t^3\left(\dfrac{1}{2}t^2 + \dfrac{1}{2}(t^2)^3\right) = t^3\left(\dfrac{1}{2}t^2 + \dfrac{1}{2}t^6\right) = \dfrac{1}{2}t^5 + \dfrac{1}{2}t^9$

c $G_Y(t) = t^{-5} G_X\left(t^4\right) = t^{-5}\left(\dfrac{1}{2}t^4 + \dfrac{1}{2}\left(t^4\right)^3\right) = t^{-5}\left(\dfrac{1}{2}t^4 + \dfrac{1}{2}t^{12}\right) = \dfrac{1}{2}t^{-1} + \dfrac{1}{2}t^7$

9 a Coefficients in the generating function must sum to 1, so $\dfrac{1}{6} + \dfrac{1}{6} + k = 1 \Rightarrow k = \dfrac{2}{3}$

b $G'_X(t) = \dfrac{1}{6} + \dfrac{1}{3}t + 3kt^2 = \dfrac{1}{6} + \dfrac{1}{3}t + 2t^2$

$\text{E}(X) = G'_X(1) = \dfrac{1}{6} + \dfrac{1}{3} + 2 = 2.5$

9 c $\mathrm{P}(X=1)=\frac{1}{6}, \mathrm{P}(X=2)=\frac{1}{6}, \mathrm{P}(X=3)=\frac{2}{3}$

So Y has the following probabilities:

$\mathrm{P}(Y=1)=\frac{1}{6}, \mathrm{P}(Y=3)=\frac{1}{6}, \mathrm{P}(Y=5)=\frac{2}{3}$

Hence the probability generating function of Y is

$\mathrm{G}_Y(t)=\frac{1}{6}t+\frac{1}{6}t^3+\frac{2}{3}t^5$

d $\mathrm{G}_Y'(t)=\frac{1}{6}+\frac{1}{2}t^2+\frac{10}{3}t^4$

$\mathrm{E}(Y)=\mathrm{G}_Y'(1)=\frac{1}{6}+\frac{1}{2}+\frac{10}{3}=\frac{24}{6}=4$

$2\mathrm{E}(X)-1=2\times 2.5-1=4=\mathrm{E}(Y)$

Challenge

1 Let $\mathrm{G}_X(t)=i_0+i_1t+i_2t^2+i_3t^3+\dots$

$\mathrm{G}_X'(t)=i_1+2i_2t+3i_3t^2+\dots$

So $\mathrm{E}(X)=\mathrm{G}_X'(1)=i_1+2i_2+3i_3+\dots$

$\mathrm{G}_Y(t)=t^b\,\mathrm{G}_X(t^a)=t^bi_0+t^bi_1t^a+t^bi_2t^{2a}+t^bi_3t^{3a}+\dots=i_0t^b+i_1t^{a+b}+i_2t^{2a+b}+i_3t^{3a+b}+\dots$

$\mathrm{G}_Y'(t)=bi_0t^{b-1}+(a+b)i_1t^{a+b-1}+(2a+b)i_2t^{2a+b-1}+(3a+b)i_3t^{3a+b-1}+\dots$

$=b(i_0t^{b-1}+i_1t^{a+b-1}+i_2t^{2a+b-1}+i_3t^{3a+b-1}+\dots)+ai_1t^{a+b-1}+2ai_2t^{2a+b-1}+3ai_3t^{3a+b-1}\dots$

$\mathrm{G}_Y'(1)=b(i_0+i_1+i_2+i_3\dots)+ai_1+2ai_2+3ai_3+\dots$

$=b(i_0+i_1+i_2+i_3\dots)+a(i_1+2i_2+3i_3+\dots)$

$=b\,\mathrm{G}_X(1)+a\,\mathrm{G}_X'(1)$

As $\mathrm{G}_X(1)=1$ and $\mathrm{G}_X'(1)=\mathrm{E}(X)$ this gives

$\mathrm{E}(Y)=\mathrm{G}_Y'(1)=b+a\,\mathrm{E}(X)=a\,\mathrm{E}(X)+b$ as required

2 a Let the random variable X denote the number of shots Holly needs to hit her first bullseye, so $X\sim\mathrm{Geo}(0.6)$

Using the standard formula $\mathrm{G}_X(t)=\frac{0.6t}{1-(1-0.6)t}=\frac{0.6t}{1-0.4t}=\frac{3t}{5-2t}$

b Let the random variable Y denote the number of shots Holly needs to hit two bullseyes, so $Y\sim$ Negative $\mathrm{B}(2,0.6)$

Using the standard formula $\mathrm{G}_Y(t)=\left(\frac{0.6t}{1-(1-0.6)t}\right)^2=\left(\frac{0.6t}{1-0.4t}\right)^2=\left(\frac{3t}{5-2t}\right)^2=\left(\mathrm{G}_X(t)\right)^2$

c Let the random variable Z denote the number of shots Holly needs to hit four bullseyes, so $Z\sim$ Negative $\mathrm{B}(4,0.6)$

Using the standard formula $\mathrm{G}_Z(t)=\left(\frac{0.6t}{1-(1-0.6)t}\right)^4=\left(\frac{0.6t}{1-0.4t}\right)^4=\left(\frac{3t}{5-2t}\right)^4=\left(\mathrm{G}_X(t)\right)^4$