Probability generating functions 7C

1 $G'_X(t) = \frac{1}{4} + \frac{1}{2}t$

So $G'_X(1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

Hence, $E(X) = G'_X(1) = \frac{3}{4}$

$G''_X(t) = \frac{1}{2}$

Hence, $\text{Var}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 = \frac{1}{2} + \frac{3}{4} - \frac{9}{16} = \frac{20}{16} - \frac{9}{16} = \frac{11}{16}$

2 $G'_X(t) = \frac{1}{6} + \frac{2}{3}t + t^2$

So $G'_X(1) = \frac{1}{6} + \frac{2}{3} + 1 = \frac{11}{6}$

Hence, mean $= E(X) = G'_X(1) = \frac{11}{6}$

$G''_X(t) = \frac{2}{3} + 2t$

$G''_X(1) = \frac{2}{3} + 2 = \frac{8}{3}$

Hence, $\text{Var}(X) = G''_X(1) + G'_X(1) - \left(G'_X(1)\right)^2 = \frac{8}{3} + \frac{11}{6} - \frac{121}{36} = \frac{162}{36} - \frac{121}{36} = \frac{41}{36}$

Standard deviation of $X = \sqrt{\text{Var}(X)} = \frac{\sqrt{41}}{6} = 1.067$ (3 d.p.)

3 a Unbiased coins have P(tail) = P(head) = $\frac{1}{2}$ So $X \sim B(3, 0.5)$ and it has this probability distribution:

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Hence the probability generating function is:

$$G_X(t) = \sum P(X = x)t^x$$
$$= \frac{1}{8}(1 + 3t + 3t^2 + t^3)$$
$$= (0.5)^3(1+t)^3 = (0.5 + 0.5t)^3$$

3 b $G'_X(t) = \frac{3}{8} + \frac{6}{8}t + \frac{3}{8}t^2$

So $G'_X(1) = \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = \frac{3}{2}$

Hence, mean $= E(X) = G'_X(1) = \frac{3}{2}$

$G''_X(t) = \frac{6}{8} + \frac{6}{8}t$

$G''_X(1) = \frac{6}{8} + \frac{6}{8} = \frac{12}{8} = \frac{3}{2}$

Hence, $\text{Var}(X) = G''_X(1) + G'_X(1) - \left(G'_X(1)\right)^2 = \frac{3}{2} + \frac{3}{2} - \frac{9}{4} = \frac{12}{4} - \frac{9}{4} = \frac{3}{4}$

4 a The random variable X representing the number of heads in four spins has a binomial distribution, $X \sim B(4, 0.6)$

So using the standard formula for the probability generating function of a discrete random variable that has a binomial distribution:
$G_X(t) = (1 - 0.6 + 0.6t)^4 = (0.4 + 0.6t)^4$

Alternatively, from first principles:

$G_X(t) = \sum P(X = x)t^x = \sum \binom{4}{x}(0.6)^x(0.4)^{4-x}t^x$

$= \sum \binom{4}{x}(0.6t)^x(0.4)^{4-x}$ (this is a binomial expansion)

$= (0.4 + 0.6t)^4$

b i Using the chain rule, $G'_X(t) = 4 \times 0.6(0.4 + 0.6t)^3 = 2.4(0.4 + 0.6t)^3$

So $G'_X(1) = 2.4(0.4 + 0.6)^3 = 2.4$

Hence, mean $= E(X) = G'_X(1) = 2.4$

ii Again using the chain rule, $G''_X(t) = 3 \times 2.4 \times 0.6(0.4 + 0.6t)^2 = 4.32(0.4 + 0.6t)^2$

$G''_X(1) = 4.32$

Hence, $\text{Var}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 = 4.32 + 2.4 - 5.76 = 0.96$

Standard deviation of $X = \sqrt{\text{Var}(X)} = \sqrt{0.96} = \frac{\sqrt{96}}{10} = \frac{\sqrt{24}}{5} = \frac{2\sqrt{6}}{5} = 0.980$ (3 d.p.)

5 Using the chain and product rules,

$$G'_X(t) = \frac{2t(2+t)^4 + 4t^2(2+t)^3}{81} = \frac{2t(2+t)^3(2+t+2t)}{81} = \frac{2t(2+t)^3(2+3t)}{81}$$

$$G'_X(1) = \frac{2(3)^3(5)}{81} = \frac{270}{81} = \frac{10}{3}$$

Hence, mean $= E(X) = G'_X(1) = \frac{10}{3}$

$$G''_X(t) = \frac{2(2+t)^3(2+3t) + 6t(2+t)^2(2+3t) + 6t(2+t)^3}{81}$$

$$= \frac{2(2+t)^2((2+t)(2+3t) + 3t(2+3t) + 3t(2+t))}{81}$$

$$= \frac{2(2+t)^2(4+8t+3t^2+6t+9t^2+6t+3t^2)}{81}$$

$$= \frac{2(2+t)^2(4+20t+15t^2)}{81}$$

$$G''_X(1) = \frac{2(3)^2(39)}{81} = \frac{702}{81} = \frac{26}{3}$$

Hence, $\text{Var}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 = \frac{26}{3} + \frac{10}{3} - \frac{100}{9} = \frac{108}{9} - \frac{100}{9} = \frac{8}{9}$

6 a $G_X(t) = \frac{9}{(4-t)^2} = 9(4-t)^{-2}$

$$G'_X(t) = 9(-2)(-1)(4-t)^{-3} = \frac{18}{(4-t)^3}$$

$$G'_X(1) = \frac{18}{27} = \frac{2}{3}$$

Hence, mean $= E(X) = G'_X(1) = \frac{2}{3}$

$$G''_X(t) = 18(-3)(-1)(4-t)^{-4} = \frac{54}{(4-t)^4}$$

$$G''_X(1) = \frac{54}{81} = \frac{2}{3}$$

Hence, $\text{Var}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 = \frac{2}{3} + \frac{2}{3} - \frac{4}{9} = \frac{8}{9}$

Standard deviation of $X = \sqrt{\text{Var}(X)} = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3} = 0.943$ (3 d.p.)

b i $G_X(t) = \sum P(X=x)t^x$, so $G_X(0) = P(X=0)$

So $P(X=0) = G_X(0) = \frac{9}{4^2} = \frac{9}{16}$

ii $G_X(t) = \sum P(X=x)t^x$, so $G'_X(t) = \sum xP(X=x)t^{x-1} \Rightarrow G'_X(0) = P(X=1)$

$$P(X=1) = G'_X(0) = \frac{18}{4^3} = \frac{18}{64} = \frac{9}{32}$$

7 a Using the chain rule and product rule to compute $G'_Y(t)$ and $G''_Y(t)$

$G'_Y(t) = 2t\,e^{t^2-1}$

Hence, mean $= E(Y) = G'_Y(1) = 2$

$G''_Y(t) = 2e^{t^2-1} + 4t^2 e^{t^2-1} = 2(1+2t^2)e^{t^2-1}$

$G''_X(1) = 6$

Hence, $Var(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 = 6 + 2 - 2^2 = 4$

b Using the general result $G_Y^{(n)}(0) = n!P(X = n)$

i $P(Y = 0) = G_Y(0) = e^{-1} = \dfrac{1}{e}$

ii $P(Y = 2) = \dfrac{1}{2!}G''_Y(0) = \dfrac{2e^{-1}}{2} = \dfrac{1}{e}$

iii $G'''_Y(t) = 8t\,e^{t^2-1} + 4t(1+2t^2)e^{t^2-1} = 4t\,e^{t^2-1}(2+1+2t^2) = 4t(3+2t^2)e^{t^2-1}$

$P(Y = 3) = \dfrac{1}{3!}G'''_Y(0) = \dfrac{1}{3!} \times 0 = 0$

iv $G_Y^{(4)}(t) = (12+24t^2)e^{t^2-1} + 8t^2(3+2t^2)e^{t^2-1} = 4(3+12t^2+4t^4)e^{t^2-1}$

$P(Y = 4) = \dfrac{1}{4!}G_Y^{(4)}(0) = \dfrac{12e^{-1}}{4!} = \dfrac{1}{2e}$

8 a At each go, probability of drawing the yellow counter is $\dfrac{1}{6}$, so $X \sim \text{Geo}\left(\dfrac{1}{6}\right)$

b From the properties of the geometric distribution $P(X = x) = \dfrac{1}{6}\left(\dfrac{5}{6}\right)^{x-1}$, so

$$G_X(t) = \sum_{x=1}^{\infty}\frac{1}{6}\left(\frac{5}{6}\right)^{x-1}t^x$$

$$= \frac{1}{6}t\sum_{x=0}^{\infty}\left(\frac{5}{6}t\right)^x \quad \text{(the sum is a geometric series)}$$

$$= \frac{1}{6}t\frac{1}{1-\frac{5}{6}t} \quad \text{(using the formula for the sum of a convergent geometric series)}$$

$$= \frac{\frac{1}{6}t}{1-\frac{5}{6}t} = \frac{t}{6-5t} = t(6-5t)^{-1}$$

c i $G'_X(t) = (6-5t)^{-1} + (-1)(-5)t(6-5t)^{-2} = (6-5t)^{-2}(6-5t+5t) = 6(6-5t)^{-2}$

$G'_X(1) = 6$; so mean $= E(X) = G'_X(1) = 6$

ii $G''_X(t) = 6(-2)(-5)(6-5t)^{-3} = 60(6-5t)^{-3}$

$G''_X(1) = 60$

Hence, $Var(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 = 60 + 6 - 6^2 = 66 - 36 = 30$

9 If $X \sim \mathrm{B}(n, p)$, then $\mathrm{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

So $\mathrm{G}_X(t) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} t^x = \sum_{x=1}^{n} \binom{n}{x} (pt)^x (1-p)^{n-x} = (1-p+pt)^n$

$\mathrm{G}'_X(t) = np(1-p+pt)^{n-1}$

$\mathrm{G}'_X(1) = np(1-p+p)^{n-1} = np$

So $\mathrm{E}(X) = \mathrm{G}'_X(1) = np$

$\mathrm{G}''_X(t) = n(n-1)p^2(1-p+pt)^{n-2}$

$\mathrm{G}''_X(1) = n(n-1)p^2(1-p+p)^{n-2} = n(n-1)p^2$

So $\mathrm{Var}(X) = \mathrm{G}''_X(1) + \mathrm{G}'_X(1) - (\mathrm{G}'_X(1))^2 = n(n-1)p^2 + np - (np)^2$

$= n^2p^2 - np^2 + np - n^2p^2 = np(1-p)$

10 If $X \sim \mathrm{Po}(\lambda)$, then $\mathrm{P}(X = x) = \dfrac{\mathrm{e}^{-\lambda}\lambda^x}{x!}$

So $\mathrm{G}_X(t) = \sum_{x=0}^{\infty} \dfrac{\mathrm{e}^{-\lambda}\lambda^x}{x!} t^x = \mathrm{e}^{-\lambda} \sum_{x=0}^{\infty} \dfrac{(\lambda t)^x}{x!} = \mathrm{e}^{-\lambda}\mathrm{e}^{\lambda t} = \mathrm{e}^{\lambda(t-1)}$

$\mathrm{G}'_X(t) = \lambda \mathrm{e}^{\lambda(t-1)}$

So $\mathrm{G}'_X(1) = \mathrm{E}(X) = \lambda$

$\mathrm{G}''_X(t) = \lambda^2 \mathrm{e}^{\lambda(t-1)}$

$\mathrm{G}''_X(1) = \lambda^2$

So $\mathrm{Var}(X) = \mathrm{G}''_X(1) + \mathrm{G}'_X(1) - (\mathrm{G}'_X(1))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

11 a Let the random variable X represent the number of calls received in a single 15-minute period. Then $X \sim \mathrm{Po}(4)$.

b Using the standard formula for the probability generating function of a Poisson distribution with $\lambda = 4$, then $\mathrm{G}_X(t) = \mathrm{e}^{4(t-1)}$

c i $\mathrm{G}'_X(t) = 4\mathrm{e}^{4(t-1)}$, so mean is $\mathrm{E}(X) = \mathrm{G}'_X(1) = 4$

ii $\mathrm{G}''_X(t) = 16\mathrm{e}^{4(t-1)}$, $\mathrm{G}''_X(1) = 16$

$\mathrm{Var}(X) = \mathrm{G}''_X(1) + \mathrm{G}'_X(1) - (\mathrm{G}'_X(1))^2 = 16 + 4 - 4^2 = 4$

Standard deviation $= \sqrt{\mathrm{Var}(X)} = \sqrt{4} = 2$

12 $G_X(1)=1$, so $a+b+c=1$ **(1)**

$G'_X(t)=b+2ct$, so $G'_X(1)=b+2c$

$E(X)=G'_X(1)$, so $b+2c=\frac{5}{4}$ **(2)**

$G''_X(t)=2c$, so $G'_X(1)=2c$

$$\text{Var}(X)=G''_X(1)+G'_X(1)-\left(G'_X(1)\right)^2=2c+\frac{5}{4}-\left(\frac{5}{4}\right)^2=2c-\frac{5}{16}$$

$$\text{So } 2c-\frac{5}{16}=\frac{7}{16}\Rightarrow c=\frac{6}{16}=\frac{3}{8}$$

Substituting the value of c into equation **(2)** gives

$$b+\frac{6}{8}=\frac{5}{4}\Rightarrow b=\frac{1}{2}$$

Substituting the values of b and c into equation **(1)** gives

$$a+\frac{1}{2}+\frac{3}{8}=1\Rightarrow a=\frac{1}{8}$$

13 $G_X(1)=1$, so $\frac{a}{b-1}=1$

$\Rightarrow a=b-1$ **(1)**

Using the quotient rule,

$$G'_X(t)=\frac{\left(b-t^2\right)a-at(-2t)}{\left(b-t^2\right)^2}=\frac{ab-at^2+2at^2}{\left(b-t^2\right)^2}=\frac{ab+at^2}{\left(b-t^2\right)^2}$$

$G'_X(1)=\frac{ab+a}{(b-1)^2}=1.5$ **(2)**

Substituting for a in equation **(2)** gives

$(b-1)b+b-1=1.5(b-1)^2$

$\Rightarrow 2b^2-2=3b^2-6b+3$

$\Rightarrow b^2-6b+5=0$

$\Rightarrow (b-5)(b-1)=0$

If $b=1$, then from equation **(1)** $a=0$, which is not a solution as a is not positive

If $b=5$, then from equation **(1)**, $a=4$

Solution $a=4, b=5$

14 a $G_X(1)=1$, so $G_X(1)=k2^4=1\Rightarrow k=\frac{1}{2^4}=\frac{1}{16}$

b $G_X(t)=\frac{1}{16}(1+t)^4=\frac{1}{2^4}(1+t)^4=(0.5)^4(1+t)^4=(0.5+0.5t)^4$

This can be written as $G_X(t)=(1-0.5+0.5t)^4$, which is the standard probability generating function for the binomial distribution $G_X(t)=(1-p+pt)^n$, with $p=0.5$ and $n=4$

So $X\sim B(4, 0.5)$

14 c $G_X(t) = E(t^X) = \sum P(X = x)t^x$

$G'_X(t) = \sum P(X = x)xt^{x-1}$

$\Rightarrow G'_X(t) = E(Xt^{X-1})$, so $G'_X(1) = E(X)$

In this case $G'_X(t) = \frac{4}{16}(1+t)^3 = \frac{1}{4}(1+t)^3$

So $E(X) = G'_X(1) = \frac{1}{4}(2)^3 = 2$

$G''_X(t) = \sum P(X = x)x(x-1)t^{x-2} = E(X(X-1)t^{x-2})$

So $G''_X(1) = E(X(X-1)) = E(X^2) - E(X)$

$\Rightarrow E(X^2) = G''_X(1) + E(X) = G''_X(1) + G'_X(1)$

So $\text{Var}(X) = E(X^2) - (E(X))^2 = G''_X(1) + G'_X(1) - (G'_X(1))^2$

In this case $G''_X(t) = \frac{3}{4}(1+t)^2$, so $G''_X(1) = 3$

So $\text{Var}(X) = 3 + 2 - 2^2 = 1$

15 a Let X be the random variable denoting the smaller of the two numbers from a roll of two fair dice. Use a sample space diagram to find the possible outcomes.

	1	**2**	**3**	**4**	**5**	**6**
1	1	1	1	1	1	1
2	1	2	2	2	2	2
3	1	2	3	3	3	3
4	1	2	3	4	4	4
5	1	2	3	4	5	5
6	1	2	3	4	5	6

From the sample space diagram, it is easy to write the probability distribution:

x	1	2	3	4	5	6
P($X = x$)	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

So the probability generating function of X can be written as

$G_X(t) = \frac{1}{36}(11t + 9t^2 + 7t^3 + 5t^4 + 3t^5 + t^6)$

b i $G'_X(t) = \frac{1}{36}(11 + 18t + 21t^2 + 20t^3 + 15t^4 + 6t^5)$

$E(X) = G'_X(\mathbf{1}) = \frac{1}{36}(11 + 18 + 21 + 20 + 15 + 6) = \frac{91}{36} = 2.528$ (3 d.p.)

15 b ii $G''_X(t) = \frac{1}{36}(18 + 42t + 60t^2 + 60t^3 + 30t^4)$

$G''(1) = \frac{1}{36}(18 + 42 + 60 + 60 + 30) = \frac{210}{36}$

Hence, $\text{Var}(X) = G''_X(1) + G'_X(1) - (G'_X(1))^2 = \frac{210}{36} + \frac{91}{36} - \frac{8281}{1296} = \frac{2555}{1296}$

Standard deviation of $X = \sqrt{\text{Var}(X)} = \frac{\sqrt{2555}}{\sqrt{1296}} = \frac{\sqrt{2555}}{36} = 1.404$ (3 d.p.)

Challenge

a $P(X = 0) = G_X(0)$, so $P(X = 0) = 0$

b Using the quotient rule,

$G'_X(t) = \frac{2 - t^k - t(-kt^{k-1})}{(2 - t^k)^2} = \frac{2 - t^k + kt^k}{(2 - t^k)^2}$

$G'_X(1) = \frac{2 - 1 + k}{(2 - 1)^2} = k + 1$

So $E(X) = G'_X(1) = k + 1$

c $P(X = 1) = G'_X(0)$

So $P(X = 1) = G'_X(0) = \frac{2}{2^2} = \frac{1}{2}$