

Probability generating functions 7B

1 a If $X \sim B(n, p)$, then $G_X(t) = (1 - p + pt)^n$

So $X \sim B(4, 0.5)$, $G_X(t) = (1 - 0.5 + 0.5t)^4 = (0.5 + 0.5t)^4 = 0.5^4(1+t)^4 = \frac{1}{16}(1+t)^4$

b $Y \sim B(6, 0.2)$, $G_Y(t) = (1 - 0.2 + 0.2t)^6 = (0.8 + 0.2t)^6$

c $X \sim B(5, 0.9)$, $G_X(t) = (1 - 0.9 + 0.9t)^5 = (0.1 + 0.9t)^5$

d If $X \sim \text{Po}(\lambda)$, then $G_X(t) = e^{\lambda(t-1)}$

$X \sim \text{Po}(3)$, $G_X(t) = e^{3(t-1)}$

e $X \sim \text{Po}(1.7)$, $G_X(t) = e^{1.7(t-1)}$

f $Y \sim \text{Po}(0.2)$, $G_Y(t) = e^{0.2(t-1)}$

2 a If $X \sim \text{Geo}(p)$, then $G_X(t) = \frac{pt}{1 - (1-p)t}$

$X \sim \text{Geo}(0.3)$, $G_X(t) = \frac{0.3t}{1 - (1-0.3)t} = \frac{0.3t}{1 - 0.7t} = \frac{3t}{10 - 7t}$

b $Y \sim \text{Geo}(0.8)$, $G_Y(t) = \frac{0.8t}{1 - (1-0.8)t} = \frac{0.8t}{1 - 0.2t} = \frac{4t}{5 - t}$

c If $X \sim \text{Negative B}(r, p)$, then $G_X(t) = \left(\frac{pt}{1 - (1-p)t} \right)^r$

$X \sim \text{Negative B}(3, 0.4)$, $G_X(t) = \left(\frac{0.4t}{1 - (1-0.4)t} \right)^3 = \left(\frac{0.4t}{1 - 0.6t} \right)^3 = \left(\frac{2t}{5 - 3t} \right)^3$

d $Y \sim \text{Negative B}(5, 0.9)$, $G_Y(t) = \left(\frac{0.9t}{1 - (1-0.9)t} \right)^5 = \left(\frac{0.9t}{1 - 0.1t} \right)^5 = \left(\frac{9t}{10 - t} \right)^5$

3 a Let the random variable X be the number of sixes obtained in 5 rolls, then $X \sim B(5, 0.2)$

So $G_X(t) = (1 - 0.2 + 0.2t)^5 = (0.8 + 0.2t)^5$

b Let the random variable Y be the number of throws of the dice until a six is thrown, then $Y \sim \text{Geo}(0.2)$

So $G_Y(t) = \frac{0.2t}{1 - (1-0.2)t} = \frac{0.2t}{1 - 0.8t} = \frac{t}{5 - 4t}$

- 3 c Let the random variable Y be the number of throws of the dice until two sixes have been thrown, then $Z \sim \text{Negative B}(2, 0.2)$

$$\text{So } G_Z(t) = \left(\frac{0.2t}{1 - (1 - 0.2)t} \right)^2 = \left(\frac{0.2t}{1 - 0.8t} \right)^2 = \left(\frac{t}{5 - 4t} \right)^2$$

- 4 a $X \sim \text{Po}(0.3)$

b $P(X = 1) = \frac{e^{-0.3} 0.3^1}{1!} = 0.3e^{-0.3} = 0.2222$ (4 d.p.)

c If $X \sim \text{Po}(\lambda)$, then $G_X(t) = e^{\lambda(t-1)}$
 So $G_X(t) = e^{0.3(t-1)}$

- 5 a $X \sim \text{Geo}(0.35)$

b $P(X = 6) = 0.35(1 - 0.35)^5 = 0.35 \times 0.65^5 = 0.0406$ (4 d.p.)

c $G_X(t) = \frac{0.35t}{1 - (1 - 0.35)t} = \frac{0.35t}{1 - 0.65t} = \frac{7t}{20 - 13t}$

- 6 $X \sim \text{B}(4, 0.8)$, so $P(X = x) = \binom{4}{x} 0.8^x (1 - 0.8)^{4-x}$

$$\begin{aligned} G_X(t) &= \sum t^x P(X = x) \\ &= (0.2)^4 + 4(0.8)(0.2)^3 t + 6(0.8)^2 (0.2)^2 t^2 + 4(0.8)^3 (0.2) t^3 + (0.8)^4 t^4 \\ &= (0.2)^4 + 4(0.2)^3 (0.8t) + 6(0.2)^2 (0.8t)^2 + 4(0.2)(0.8t)^3 + (0.8t)^4 \\ &= (0.2 + 0.8t)^4 \end{aligned}$$

- 7 $X \sim \text{Po}(3.5)$, so $P(X = x) = \frac{e^{-3.5} 3.5^x}{x!}$

$$\begin{aligned} G_X(t) &= \sum t^x P(X = x) = \sum t^x \frac{e^{-3.5} 3.5^x}{x!} \\ &= e^{-3.5} \sum \frac{t^x 3.5^x}{x!} = e^{-3.5} \sum \frac{(3.5t)^x}{x!} \\ &= e^{-3.5} \left(1 + 3.5t + \frac{(3.5t)^2}{2!} + \frac{(3.5t)^3}{3!} + \dots \right) \end{aligned}$$

The expression in brackets is the Maclaurin expansion of e^x where $x = 3.5t$, so:

$$G_X(t) = e^{-3.5} e^{3.5t} = e^{3.5t - 3.5} = e^{3.5(t-1)}$$

$$8 \quad Y \sim \text{Geo}(0.7), \text{ so } P(Y = y) = (1 - 0.7)^{y-1} 0.7$$

$$\begin{aligned} G_Y(t) &= \sum_{y=1}^{\infty} t^y P(Y = y) = \sum_{y=1}^{\infty} t^y 0.3^{y-1} 0.7 \\ &= 0.7t \sum_{y=1}^{\infty} (0.3t)^{y-1} = 0.7t \sum_{y=0}^{\infty} (0.3t)^y \\ &= 0.7t(1 + 0.3t + (0.3t)^2 + \dots) \end{aligned}$$

The bracketed expression is the sum of an infinite geometric series, with first term 1 and common ratio $0.3t$. Using the formula for the sum of a geometric series gives:

$$(1 + 0.3t + (0.3t)^2 + \dots) = \frac{1}{1 - 0.3t}$$

$$\text{So } G_Y(t) = 0.7t \frac{1}{1 - 0.3t} = \frac{0.7t}{1 - 0.3t}$$

$$9 \quad X \sim B(n, p), \text{ so } P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

$$\begin{aligned} G_X(t) &= \sum t^x P(X = x) = \sum t^x \binom{n}{x} p^x (1 - p)^{n-x} \\ &= \sum \binom{n}{x} (pt)^x (1 - p)^{n-x} \end{aligned}$$

$$= (1 - p)^n + \binom{n}{1} (1 - p)^{n-1} pt + \binom{n}{2} (1 - p)^{n-2} (pt)^2 + \binom{n}{3} (1 - p)^{n-3} (pt)^3 + \dots + (pt)^n$$

This is the binomial expansion of $(a + b)^n$ (see Pure Year 1, Chapter 8) with $a = 1 - p$ and $b = pt$

$$\text{So } G_X(t) = (1 - p + pt)^n$$

$$10 \quad X \sim \text{Po}(\lambda), \text{ so } P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\begin{aligned} G_X(t) &= \sum t^x P(X = x) = \sum t^x \frac{e^{-\lambda} \lambda^x}{x!} \\ &= e^{-\lambda} \sum \frac{(\lambda t)^x}{x!} = e^{-\lambda} \left(1 + \lambda t + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \dots \right) \end{aligned}$$

The bracketed expression is the Maclaurin expansion of e^x where $x = \lambda t$

$$\text{So } G_X(t) = e^{-\lambda} e^{\lambda t} = e^{\lambda t - \lambda} = e^{\lambda(t-1)}$$

11 $Y \sim \text{Geo}(p)$, so $P(Y = y) = p(1-p)^{y-1}$

$$\begin{aligned} G_Y(t) &= \sum_{y=1}^{\infty} t^y P(Y = y) = \sum_{y=1}^{\infty} t^y p(1-p)^{y-1} \\ &= pt \sum_{y=1}^{\infty} (t(1-p))^{y-1} = pt \sum_{y=0}^{\infty} (t(1-p))^y \\ &= pt(1 + (1-p)t + ((1-p)t)^2 + \dots) \end{aligned}$$

The bracketed expression is the sum of an infinite geometric series, with first term 1 and common ratio $(1-p)t$. Using the formula for the sum of a geometric series gives:

$$\text{So } G_Y(t) = pt \frac{1}{1-(1-p)t} = \frac{pt}{1-(1-p)t}$$