Probability generating functions 7A

1 \( G_X(t) = 0.3 + 0.2t + 0.5t^2 \)

a The powers of \( t \) in the probability generating function of \( X \) correspond to the sample space of \( X \). In this case, the sample space is therefore 0, 1, 2

b i \( P(X = 0) \) is the coefficient of \( t^0 \) in \( G_X(t) \), so \( P(X = 0) = 0.3 \)

ii Note that the sample space is 0, 1, 2, i.e. \( X \) does not take values less than 0, so \( P(X \geq 0) = 1 \)

2 \( G_X(t) = \frac{1}{8}(1 + t)^3 = \frac{1}{8} + \frac{3}{8}t + \frac{3}{8}t^2 + \frac{1}{8}t^3 \)

a The powers of \( t \) in the probability generating function of \( X \) correspond to the sample space of \( X \). In this case, the sample space is therefore 0, 1, 2, 3

b i \( P(X = 1) \) is the coefficient of \( t^1 \) in \( G_X(t) \), so \( P(X = 1) = \frac{3}{8} \)

ii Using the fact that the coefficients of \( t^x \) in the function \( G_X(t) \) are the probabilities \( P(X = x) \):

\[
P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}
\]

Alternatively derive the result from

\[
P(X \leq 2) = 1 - P(X = 3) = 1 - \frac{1}{8} = \frac{7}{8}
\]

3 \( G_Y(t) = 0.7 + 0.1(t^2 + t^3 + t^5) \)

a There is no \( t^1 \) term in this probability generating function, so \( P(Y = 1) = 0 \)

b Using the fact that the coefficients of \( t^x \) in the function \( G_Y(t) \) are the probabilities \( P(Y = x) \):

\[
P(Y < 3) = P(Y = 0) + P(Y = 1) + P(Y = 2) = 0.7 + 0.1 = 0.8
\]

c \( P(3 \leq Y \leq 6) = P(Y = 3) + P(Y = 4) + P(Y = 5) + P(Y = 6) = 0.1 + 0.1 = 0.2 \)

4 It is a fair dice, so \( P(X = i) = \frac{1}{6} \) for \( i = 1, 2, 3, 4, 5, 6 \)

So \( G_X(t) = \frac{1}{6}(t + t^2 + t^3 + t^4 + t^5 + t^6) \)
5  Let \( X \) be the random variable representing the outcome of a tetrahedral (4-sided) dice. So \( X \) can take the values 1, 2, 3, 4

\[
P(X = 1) = 0.4
\]

\[
P(X = 2) = P(X = 3) = P(X = 4)
\]

And probabilities must sum to 1, so

\[
P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 1
\]

Therefore \( 0.4 + 3P(X = 2) = 1 \Rightarrow 3P(X = 2) = 0.6 \Rightarrow P(X = 2) = 0.2, \ P(X = 3) = 0.2, \ P(X = 4) = 0.2 \)

Hence the probability generating function can be written as:

\[
G_X(t) = 0.4t + 0.2t^2 + 0.2t^3 + 0.2t^4
\]

Alternatively, the equation can be written in fractional form:

\[
G_X(t) = \frac{2}{5}t + \frac{1}{5}t^2 + \frac{1}{5}t^3 + \frac{1}{5}(2t^2 + t^3 + t^4)
\]

6 a \( P(X = 1) = \frac{1}{10}, \ P(X = 2) = \frac{2}{10}, \ P(X = 3) = \frac{3}{10}, \ P(X = 4) = \frac{4}{10} \)

So the probability generating function is:

\[
G_X(t) = \frac{1}{10}t + 2t^2 + 3t^3 + 4t^4
\]

b \( P(X = 1) = \frac{1}{14}, \ P(X = 2) = \frac{4}{14}, \ P(X = 3) = \frac{9}{14} \)

So the probability generating function is:

\[
G_X(t) = \frac{1}{14}(t + 4t^2 + 9t^3)
\]

7 a Use the fact that \( G_Y(1) = 1 \)

(This is because \( G_Y(t) = \sum P(Y = y)t^y \) and for \( t = 1 \) this gives \( G_Y(1) = \sum P(Y = y) = 1 \))

\[
G_Y(1) = k(2 + 1 + 1)^2 = 1
\]

\[
\Rightarrow 16k = 1
\]

So \( k = \frac{1}{16} = 0.0625 \)

b \( P(Y = 1) \) is find the coefficient of the \( t \) term in the probability function

\[
G_Y(t) = \frac{1}{16}(2 + t + t^2)^2
\]

\[
= \frac{1}{16}(2 + 4t + 5t^2 + 2t^3 + t^4)
\]

So \( P(Y = 1) = \frac{4}{16} = \frac{1}{4} = 0.25 \)
8 a To calculate $k$, use $G_X(1) = 1$ Thus

$$k(1 + 2 + 2)^2 = 1$$

$\Rightarrow 25k = 1$

So $k = \frac{1}{25}$

b Expand $G_X(t)$:

$$G_X(t) = \frac{1}{25} (1 + 2t + 2t^2)(1 + 2t + 2t^2)$$

$$= \frac{1}{25} (1 + 4t + 8t^2 + 8t^3 + 4t^4)$$

Using the fact that the coefficients of $t^x$ in the function $G_X(t)$ are the probabilities $P(X = x)$, the probability distribution of $X$ is:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x)$</td>
<td>$\frac{1}{25}$</td>
<td>$\frac{4}{25}$</td>
<td>$\frac{8}{25}$</td>
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9 a Let $X$ be the random variable denoting the sum of the scores of two fair four-sided dice. Use a sample space diagram to find the possible outcomes.

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<tr>
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From the sample space diagram, it is easy to find the probabilities for all possible values of $X$, i.e. 2, 3, 4, 5, 6, 7, 8, and hence write the probability distribution:

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x)$</td>
<td>$\frac{1}{16}$</td>
<td>$\frac{2}{16}$</td>
<td>$\frac{3}{16}$</td>
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<td>$\frac{1}{16}$</td>
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</tbody>
</table>

b Use part a to construct the probability generating function:

$$G_X(t) = \frac{1}{16} (t^2 + 2t^3 + 3t^4 + 4t^5 + 3t^6 + 2t^7 + t^8)$$

10 Calculate $G_X(1) = 0.1(2 + 5 + 4) = 1.1$. As $G_X(1) = 1$ for any probability generating function, and $G_X(1) \neq 1$ in this case, $G$ cannot be a probability generating function.
11  a  Use the fact that $G_Y(1) = 1$
\[
k(1+1)^{10} = 1
\]
\[
\Rightarrow 1024k = 1
\]
So $k = \frac{1}{1024}$

b  The largest power of $t$ in the probability generating function is 10, and so this is the largest value that $Y$ can take.
\[
P(Y = 10) \text{ is the coefficient of } t^{10} \text{ in the probability generating function } G_Y(t)
\]
The $t^{10}$ coefficient of $(1+t)^{10}$ is 1, so the $t^{10}$ coefficient of $k(1+t)^{10}$ is:
\[
P(Y = 10) = k \times 1 = \frac{1}{1024}
\]

c  $P(Y = 5)$ is the coefficient of $t^5$ in the probability generating function $G_Y(t)$

It is possible to find this by expanding $k(1+t)^{10}$ by hand, but it is easier to use the binomial expansion (covered in Pure Year 1, Chapter 8).

From the binomial expansion, the $t^5$ term of $(1+t)^{10}$ is $\binom{10}{5}t^5$ so:
\[
P(Y = 5) = k\binom{10}{5} = \frac{1}{1024} \times \frac{10!}{5!5!}
\]
\[
= \frac{10 \times 9 \times 8 \times 7 \times 6}{1024 \times 5 \times 4 \times 3 \times 2}
\]
\[
= \frac{9 \times 4 \times 7}{256} = \frac{63}{256}
\]

d  Generalising the result from part c
\[
P(Y = r) = k\binom{10}{r} = \frac{1}{1024}\binom{10}{r} = \left(\frac{1}{2}\right)^{10} \binom{10}{r}
\]
\[
= \binom{10}{r} \left(\frac{1}{2}\right)^{10-r} = \binom{10}{r} \left(\frac{1}{2}\right)^{10-r}
\]

This is the probability mass function of a binomial distribution (see Statistics and Mechanics Year 1, Chapter 6). So $Y$ has a binomial distribution, i.e. $Y \sim B(n, p)$

In this case, $n = 10$ and $p = 0.5$, so $Y \sim B(10, 0.5)$ or $Y \sim B\left(10, \frac{1}{2}\right)$