Probability generating functions 7A

1 $G_X(t) = 0.3 + 0.2t + 0.5t^2$

a The powers of t in the probability generating function of X correspond to the sample space of X. In this case, the sample space is therefore 0, 1, 2

b **i** $P(X = 0)$ is the coefficient of t^0 in $G_X(t)$, so $P(X = 0) = 0.3$

ii Note that the sample space is 0, 1, 2, i.e. X does not take values less than 0, so $P(X \geqslant 0) = 1$

2 $G_X(t) = \frac{1}{8}(1+t)^3 = \frac{1}{8} + \frac{3}{8}t + \frac{3}{8}t^2 + \frac{1}{8}t^3$

a The powers of t in the probability generating function of X correspond to the sample space of X. In this case, the sample space is therefore 0, 1, 2, 3

b **i** $P(X = 1)$ is the coefficient of t^1 in $G_X(t)$, so $P(X = 1) = \frac{3}{8}$

ii Using the fact that the coefficients of t^x in the function $G_X(t)$ are the probabilities $P(X = x)$:

$$\begin{aligned} P(X \leqslant 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8} \end{aligned}$$

Alternatively derive the result from

$$P(X \leqslant 2) = 1 - P(X = 3) = 1 - \frac{1}{8} = \frac{7}{8}$$

3 $G_Y(t) = 0.7 + 0.1(t^2 + t^3 + t^5)$

a There is no t^1 term in this probability generating function, so $P(Y = 1) = 0$

b Using the fact that the coefficients of t^x in the function $G_X(t)$ are the probabilities $P(X = x)$:

$$\begin{aligned} P(Y < 3) &= P(Y = 0) + P(Y = 1) + P(Y = 2) \\ &= 0.7 + 0.1 = 0.8 \end{aligned}$$

c $P(3 \leqslant Y \leqslant 6) = P(Y = 3) + P(Y = 4) + P(Y = 5) + P(Y = 6) = 0.1 + 0.1 = 0.2$

4 It is a fair dice, so $P(X = i) = \frac{1}{6}$ for $i = 1, 2, 3, 4, 5, 6$

So $G_X(t) = \frac{1}{6}(t + t^2 + t^3 + t^4 + t^5 + t^6)$

5 Let X be the random variable representing the outcome of a tetrahedral (4-sided) dice.
So X can take the values 1, 2, 3, 4

$P(X=1)=0.4$

$P(X=2)=P(X=3)=P(X=4)$

And probabilities must sum to 1, so

$P(X=1)+P(X=2)+P(X=3)+P(X=4)=1$

Therefore $0.4+3P(X=2)=1 \Rightarrow 3P(X=2)=0.6$

$\Rightarrow P(X=2)=0.2,\ P(X=3)=0.2,\ P(X=4)=0.2$

Hence the probability generating function can be written as:

$G_X(t)=0.4t+0.2t^2+0.2t^3+0.2t^4$

Alternatively, the equation can be written in fractional form:

$G_X(t)=\frac{2}{5}t+\frac{1}{5}t^2+\frac{1}{5}t^3+\frac{1}{5}t^4=\frac{1}{5}(2t+t^2+t^3+t^4)$

6 a $P(X=1)=\frac{1}{10},\ P(X=2)=\frac{2}{10},\ P(X=3)=\frac{3}{10},\ P(X=4)=\frac{4}{10}$

So the probability generating function is:

$G_X(t)=\frac{1}{10}(t+2t^2+3t^3+4t^4)$

b $P(X=1)=\frac{1}{14},\ P(X=2)=\frac{4}{14},\ P(X=3)=\frac{9}{14}$

So the probability generating function is:

$G_X(t)=\frac{1}{14}(t+4t^2+9t^3)$

7 a Use the fact that $G_Y(1)=1$

(This is because $G_Y(t)=\sum P(Y=y)t^y$ and for $t=1$ this gives $G_Y(1)=\sum P(Y=y)=1$)

$G_Y(1)=k(2+1+1)^2=1$

$\Rightarrow 16k=1$

So $k=\frac{1}{16}=0.0625$

b P(Y = 1) is find the coefficient of the t term in the probability function

$G_Y(t)=\frac{1}{16}(2+t+t^2)^2$

$=\frac{1}{16}(2+4t+5t^2+2t^3+t^4)$

So $P(Y=1)=\frac{4}{16}=\frac{1}{4}=0.25$

8 **a** To calculate k, use $G_X(1)=1$ Thus

$k(1+2+2)^2=1$

$\Rightarrow 25k=1$

So $k=\frac{1}{25}$

b Expand $G_X(t)$:

$$G_X(t)=\frac{1}{25}(1+2t+2t^2)(1+2t+2t^2)$$
$$=\frac{1}{25}(1+4t+8t^2+8t^3+4t^4)$$

Using the fact that the coefficients of t^x in the function $G_X(t)$ are the probabilities $P(X=x)$, the probability distribution of X is:

x	0	1	2	3	4
$P(X=x)$	$\frac{1}{25}$	$\frac{4}{25}$	$\frac{8}{25}$	$\frac{8}{25}$	$\frac{4}{25}$

9 **a** Let X be the random variable denoting the sum of the scores of two fair four-sided dice. Use a sample space diagram to find the possible outcomes.

	1	**2**	**3**	**4**
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

From the sample space diagram, it is easy to find the probabilities for all possible values of X, i.e. 2, 3, 4, 5, 6, 7, 8, and hence write the probability distribution:

x	2	3	4	5	6	7	8
$P(X=x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

b Use part **a** to construct the probability generating function:

$$G_X(t)=\frac{1}{16}(t^2+2t^3+3t^4+4t^5+3t^6+2t^7+t^8)$$

10 Calculate $G_X(1)=0.1(2+5+4)=1.1$. As $G_X(1)=1$ for any probability generating function, and $G_X(1)\neq 1$ in this case, G cannot be a probability generating function.

11 a Use the fact that $G_Y(1) = 1$

$k(1+1)^{10} = 1$

$\Rightarrow 1024k = 1$

So $k = \frac{1}{1024}$

b The largest power of t in the probability generating function is 10, and so this is the largest value that Y can take.

$P(Y = 10)$ is the coefficient of t^{10} in the probability generating function $G_Y(t)$

The t^{10} coefficient of $(1+t)^{10}$ is 1, so the t^{10} coefficient of $k(1+t)^{10}$ is:

$P(Y = 10) = k \times 1 = \frac{1}{1024}$

c $P(Y = 5)$ is the coefficient of t^5 in the probability generating function $G_Y(t)$

It is possible to find this by expanding $k(1+t)^{10}$ by hand, but it is easier to use the binomial expansion (covered in Pure Year 1, Chapter 8).

From the binomial expansion, the t^5 term of $(1+t)^{10}$ is $\binom{10}{5}t^5$ so:

$$P(Y = 5) = k\binom{10}{5} = \frac{1}{1024} \times \frac{10!}{5!5!}$$
$$= \frac{10 \times 9 \times 8 \times 7 \times 6}{1024 \times 5 \times 4 \times 3 \times 2}$$
$$= \frac{9 \times 4 \times 7}{1024} = \frac{9 \times 7}{256} = \frac{63}{256}$$

d Generalising the result from part **c**

$$P(Y = r) = k\binom{10}{r} = \frac{1}{1024}\binom{10}{r} = \left(\frac{1}{2}\right)^{10}\binom{10}{r}$$
$$= \binom{10}{r}\left(\frac{1}{2}\right)^{10-r+r} = \binom{10}{r}\left(\frac{1}{2}\right)^{r}\left(\frac{1}{2}\right)^{10-r}$$

This is the probability mass function of a binomial distribution (see Statistics and Mechanics Year 1, Chapter 6). So Y has a binomial distribution, i.e. $Y \sim B(n, p)$

In this case, $n = 10$ and $p = 0.5$, so $Y \sim B(10, 0.5)$ or $Y \sim B\left(10, \frac{1}{2}\right)$