Central limit theorem 5B

1 a Let X be the random discrete variable $X \sim \text{Po}(3)$ and let T denote the sum of the 10 sample observations, so $T \sim \text{Po}(10\times 3)$, i.e. $T \sim \text{Po}(30)$

If the sample mean = 2.5, then $T = 10\times 2.5 = 25$. So the probability that the sample mean is less than 25 is $\text{P}(T \leqslant 25)$

By calculation $\text{P}(T \leqslant 25) = 0.2084$ (4 d.p.)

b By the central limit theorem, $\bar{X} \approx\sim \text{N}\left(3, \frac{3}{10}\right)$, i.e. $X \approx\sim \text{N}(3, 0.3)$

Using a calculator, $\text{P}(\bar{X} \leqslant 2.5) = 0.1807$ (4 d.p.)

The two answers are not very close. This is because the estimate found in part **b** is not very accurate as the sample size is too small.

2 a $X \sim \text{Geo}(0.25)$

$$\text{E}(X) = \frac{1}{p} = \frac{1}{0.25} = 4$$

$$\text{Var}(X) = \frac{1-p}{p^2} = \frac{0.75}{0.0625} = 12$$

b By the central limit theorem, $\bar{X} \approx\sim \text{N}\left(4, \frac{12}{12}\right)$, i.e. $X \approx\sim \text{N}(4, 1)$

$\text{P}(\bar{X} > 5) = 1 - \text{P}(\bar{X} < 5) \approx 1 - 0.8413 = 0.1587$ (4 d.p.)

3 $X \sim \text{B}(10, 0.2)$

$\text{E}(X) = np = 10\times 0.2 = 2$

$\text{Var}(X) = np(1-p) = 2\times 0.8 = 1.6$

By the central limit theorem $\bar{X} \approx\sim \text{N}\left(2, \frac{1.6}{20}\right)$, i.e. $\bar{X} \approx\sim \text{N}(2, 0.8)$

$\text{P}(\bar{X} \leqslant 2.4) \approx 0.9214$ (4 d.p.)

4 a Let discrete random variable X be the number of coins flips required by a student to get 5 heads, then $X \sim \text{Negative B}(5, 0.5)$

$$\text{E}(X) = \frac{r}{p} = \frac{5}{0.5} = 10$$

b $\text{Var}(X) = \frac{r(1-p)}{p^2} = 10$

By the central limit theorem $\bar{X} \approx\sim \text{N}\left(10, \frac{10}{20}\right)$, i.e. $\bar{X} \approx\sim \text{N}(10, 0.5)$

$\text{P}(\bar{X} \leqslant 9) \approx 0.0786$ (4 d.p.)

5 a Let X be the number of thunderstorms hitting the town each month, then $X \sim \text{Po}(3)$

$$\text{P}(X=4)=\frac{e^{-3}3^4}{4!}=0.1680\ (4\text{ d.p.})$$

b $\text{E}(X)=\text{Var}(X)=3$

By the central limit theorem $\bar{X}\approx\sim \text{N}\left(3,\frac{3}{12}\right)$, i.e. $\bar{X}\approx\sim \text{N}(3, 0.25)$

$\text{P}(\bar{X}\leqslant 2.5)\approx 0.1587\ (4\text{ d.p.})$

6 a Let the discrete random variable X be the number of donors that will have to be tested before finding a match, then $X \sim \text{Geo}(0.2)$

$$\text{E}(X)=\frac{1}{p}=\frac{1}{0.2}=5$$

b $\text{Var}(X)=\frac{1-p}{p^2}=\frac{0.8}{0.2^2}=\frac{80}{4}=20$

By the central limit theorem $\bar{X}\approx\sim \text{N}\left(5,\frac{20}{20}\right)$, i.e. $\bar{X}\approx\sim \text{N}(5,1)$

$\text{P}(\bar{X}>5.5)=1-\text{P}(\bar{X}<5.5)\approx 1-0.6915=0.3085\ (4\text{ d.p.})$

7 a Let X be the number of houses that David must visit to sell 10 raffle tickets, then

$X \sim \text{Negative B}\left(10,\frac{1}{3}\right)$

$$\text{P}(X=35)=\binom{34}{9}\times\left(\frac{1}{3}\right)^{10}\times\left(\frac{2}{3}\right)^{25}=0.0352\ (4\text{ d.p.})$$

b $\text{E}(X)=\frac{r}{p}=10\times 3=30$

$\text{Var}(X)=\frac{r(1-p)}{p^2}=\frac{10\times 2\times 9}{3}=60$

By the central limit theorem $\bar{X}\approx\sim \text{N}\left(30,\frac{60}{20}\right)$, i.e. $\bar{X}\approx\sim \text{N}(30,3)$

$\text{P}(\bar{X}\leqslant 35)\approx 0.9981\ (4\text{ d.p.})$

8 a Let the discrete random variable X be the number of calls received by the telephonist in the five-minute period before her break, then $C \sim \text{Po}(10)$. Let T be the total number of calls received in this period for the 30 days the telephonist records the calls, then $T=30\bar{C}$

By the central limit theorem $\bar{C}\approx\sim \text{N}\left(10,\frac{10}{30}\right)$

$$\text{P}(T>350)=\text{P}\left(\bar{C}>\frac{350}{30}\right)=1-\text{P}\left(\bar{C}<\frac{350}{30}\right)\approx 1-0.9981=0.0019\ (4\text{ d.p.})$$

b $\text{P}(C<9)\approx 0.0416\ (4\text{ d.p.})$