Geometric and negative binomial distributions Mixed exercise 3

1 a Let X denote the number of times required to throw a multiple of 3, $X \sim \text{Geo}\left(\frac{1}{4}\right)$

$$\text{P}(X=5)=\frac{1}{4}\left(\frac{3}{4}\right)^4=\frac{81}{1024}=0.0791 \text{ (4 d.p.)}$$

b $\text{P}(X \geqslant 3)=\left(\frac{3}{4}\right)^2=\frac{9}{16}=0.5625$

2 a $X \sim \text{Geo}(0.1)$

b $\text{E}(X)=\frac{1}{0.1}=10$

$$\text{Var}(X)=\frac{1-0.1}{0.1^2}=\frac{0.9}{0.01}=90$$

c $\text{P}(X \geqslant 12)=0.9^{11}=0.3138 \text{ (4 d.p.)}$

3 a A geometric distribution model, $X \sim \text{Geo}(p)$ where p is the probability of Olivia hitting the target.

b $\text{E}(X)=\frac{1}{p}=6 \Rightarrow p=\frac{1}{6}$

$$\text{P}(X=5)=\frac{1}{6}\left(\frac{5}{6}\right)^4=\frac{625}{7776}=0.0804 \text{ (4 d.p.)}$$

c $\text{Var}(X)=\frac{1-\frac{1}{6}}{\left(\frac{1}{6}\right)^2}=\frac{5\times 36}{6}=30$

d The model assumes that the throws are independent, and the probability of hitting the target is the same for each throw.

4 $X \sim \text{Geo}(0.1)$

If x is the maximum number of times that dice is rolled, then Soujit requires $P(X \leqslant x)<0.5$

$\text{P}(X \leqslant x)=1-0.9^x$

So $1-0.9^x<0.5$

$\Rightarrow 0.9^x>0.5$

$\Rightarrow x<\frac{\log 0.5}{\log 0.9}$

$\Rightarrow x<6.579$

Solution $x=6$

5 a Let X denote the number of over-ripe avocados found in a box of 24, so $X \sim \text{B}(24, 0.02)$

$$\text{P}(X>3)=1-\text{P}(X \leqslant 3)$$
$$=1-0.998766=0.001234 \text{ (4 s.f.)}$$

b Let Y denote the first time a box is rejected, $Y \sim \text{Geo}(0.001234)$

$$\text{P}(Y=20)=0.001234(1-0.001234)^{19}=0.0012 \text{ (4 d.p.)}$$

6 a Use the model $X \sim \text{Geo}(0.2)$

$P(X = 6) = 0.2 \times 0.8^5 = 0.0655$ (4 d.p.)

b Let Y denote the number of games required to win two prizes, so $Y \sim \text{Negative B}(2, 0.2)$

$$P(Y = 10) = \binom{9}{1} \times (0.2)^2 \times (0.8)^8 = 0.0604 \text{ (4 d.p.)}$$

c Let Z denote the number of games required to win five prizes, so $Z \sim \text{Negative B}(5, 0.2)$

$$E(Z) = \frac{5}{0.2} = 25$$

$$\text{Var}(Z) = \frac{5(1-0.2)}{0.2^2} = \frac{4}{0.04} = 100$$

$$\sigma = \sqrt{\text{Var}(Z)} = \sqrt{100} = 10$$

d $X \sim \text{Negative B}(r, p)$

$$E(X) = \frac{r}{p} = 12 \Rightarrow r = 12p$$

$$\text{Var}(X) = \frac{r(1-p)}{p^2} = 16 \Rightarrow r(1-p) = 16p^2$$

Substituting for $r = 12p$ and dividing by p (as $p \neq 0$) gives

$12(1-p) = 16p$

$$\Rightarrow p = \frac{12}{28} = \frac{3}{7} = 0.4286 \text{ (4 d.p.)}$$

7 a $X \sim \text{Negative B}(4, p)$

$$\text{Var}(X) = \frac{4(1-p)}{p^2} = 15 \Rightarrow 15p^2 + 4p - 4 = 0$$

Factorising gives $(5p-2)(3p+2) = 0$

As $p > 0$, solution is $p = \frac{2}{5} = 0.4$

b $$P(X = 10) = \binom{9}{3} \times (0.4)^4 \times (0.6)^6 = 0.1003 \text{ (4 d.p.)}$$

c Let Y be the number of sixes thrown in 8 rolls of the dice, so $Y \sim B(8, 0.4)$

$P(X > 8) = P(Y \leqslant 3) = 0.5941$ (4 d.p.)

7 d As each roll is independent, this requires finding the probability that it takes 8 more rolls of the dice to throw 3 further sixes. If Z is the number of times the dice is rolled until 3 sixes have occurred, then $Z \sim \text{Negative B}(3, 0.4)$ and the required probability is $P(Z = 8)$

$$P(Z = 8) = \binom{7}{2} \times (0.4)^3 \times (0.6)^5 = 0.1045 \text{ (4 d.p.)}$$

8 a $X \sim \text{Geo}(0.65)$

$P(X = 2) = 0.65(1-0.65) = 0.65 \times 0.35 = 0.2275$

b i Let Y be the number of goals Roberta scores in 8 penalties, then $Y \sim B(8, 0.65)$

$$P(Y = 8) = \binom{8}{5} \times (0.65)^5 \times (0.35)^3 = 0.2786 \text{ (4 d.p.)}$$

8 b ii Let Z be the number of penalties taken until Roberta scores 5 goals, then

$Z \sim \text{Negative B}(5, 0.65)$

$$P(Z = 8) = \binom{7}{4} \times (0.65)^5 \times (0.35)^3 = 0.1741 \text{ (4 d.p.)}$$

iii Let M be the number of goals Roberta scores in 9 penalties, then $M \sim B(9, 0.65)$

$P(M \leqslant 4) = 0.1717$ (4 d.p.)

iv As each penalty is independent, this requires finding the probability that Roberta scores 2 more goals in the 5 penalties she takes after her first two successful attempts. If N is the number of goals Roberta scores in 5 penalties, then $N \sim B(5, 0.65)$ and the required probability is $P(N = 5)$

$$P(N = 5) = \binom{5}{2} \times (0.65)^2 \times (0.35)^3 = 0.1811 \text{ (4 d.p.)}$$

c $D \sim \text{Negative B}(5, 0.65)$

$$E(D) = \frac{5}{0.65} = \frac{100}{13} = 7.69 \text{ (2 d.p.)}$$

$$\text{Var}(D) = \frac{5(1-0.65)}{0.65^2} = \frac{1.75}{0.4225} = \frac{17500}{4225} = \frac{700}{169}$$

$$\sigma = \sqrt{\text{Var}(Z)} = \sqrt{\frac{700}{169}} = \frac{10\sqrt{7}}{13} = 2.04 \text{ (2 d.p.)}$$

d This is the probability that Sukie fails to score with her first penalty times the probability that Roberta scores her first penalty:

$$P(\text{Sukie doesn't score}) \times P(\text{Roberta scores}) = (1-0.4) \times 0.65 = 0.39$$

e $P(\text{Sukie doesn't score}) \times P(\text{Roberta doesn't score}) \times P(\text{Sukie scores})$

$$= (1-0.4) \times (1-0.65) \times 0.4 = 0.084$$

f The probability that Sukie and Roberta each fail to score from their first three penalties is:

$$(1-0.4)^3 \times (1-0.65)^3 = 0.216 \times 0.042875 = 0.009261$$

Challenge

1 a $X \sim \text{Negative B}(2, p)$

b $Y_1 \sim \text{Geo}(p),\ Y_2 \sim \text{Geo}(p)$

c $X = Y_1 + Y_2$

d The mean of X is the sum of the means of Y_1 and Y_2 so

$$E(X) = E(Y_1) + E(Y_2) = \frac{1}{p} + \frac{1}{p} = \frac{2}{p}$$

2 $X \sim \text{Negative B}(r, p)$

Let $Y_1, \ldots, Y_r$ be random variables, with $Y_1 \sim \text{Geo}(p), \ldots, Y_r \sim \text{Geo}(p)$

such that $X = Y_1 + \ldots + Y_r = \sum_{i=1}^{i=r} Y_i$

Then $E(X) = \sum_{i=1}^{i=r} E(Y_i) = \sum_{i=1}^{i=r} \frac{1}{p} = r \times \frac{1}{p} = \frac{r}{p}$

Also $\text{Var}(X) = \sum_{i=1}^{i=r} \text{Var}(Y_i) == \sum_{i=1}^{i=r} \frac{1-p}{p^2} = r \times \frac{1-p}{p^2} = \frac{r(1-p)}{p^2}$