Geometric and negative binomial distributions 3D

1 a $X \sim \text{Negative B}(3, 0.4)$

$\text{E}(X) = \frac{3}{0.4} = \frac{15}{2} = 7.5$

b $\text{Var}(X) = \frac{3(1-0.4)}{0.4^2} = \frac{1.8}{0.16} = \frac{45}{4} = 11.25$

2 a $Y \sim \text{Negative B}(10, 0.75)$

$\text{E}(Y) = \frac{10}{0.75} = \frac{40}{3} = 13.33 \text{ (2 d.p.)}$

b $\text{Var}(Y) = \frac{10(1-0.75)}{0.75^2} = \frac{2.5 \times 16}{9} = \frac{40}{9} = 4.44 \text{ (2 d.p)}$

3 a $M \sim \text{Negative B}(2, p)$

$\text{E}(M) = \frac{2}{p} = 8 \Rightarrow p = \frac{1}{4} = 0.25$

b $\text{P}(M = 5) = \binom{4}{1} \times (0.25)^2 \times (0.75)^3 = 0.1055 \text{ (4 d.p.)}$

c $\text{Var}(M) = \frac{2(1-0.25)}{0.25^2} = 1.5 \times 16 = 24$

4 a $\text{Var}(D) = \frac{8(1-p)}{p^2} = 30$

$\Rightarrow 15p^2 + 4p - 4 = 0$

Solve by factorising or by using the quadratic equation. Using the quadratic equation gives:

$p = \frac{-4 \pm \sqrt{4^2 - 4(15)(-4)}}{2(15)} = \frac{-4 \pm \sqrt{256}}{30} = \frac{-4 \pm 16}{30}$

As $p > 0$, solution is $p = \frac{12}{30} = \frac{2}{5} = 0.4$

b i $\text{P}(D = 12) = \binom{11}{7} \times (0.4)^8 \times (0.6)^4 = 0.0280 \text{ (4 d.p.)}$

ii As $D \sim \text{Negative B}(8, 0.4)$, $\text{P}(D \leq 10)$ is the probability that there will be 8 successes or more in the first 10 trials. As the first trial is successful, find the probability that there will be 7 successes or more in the next 9 trials. Let E be the number of successes in 9 trials, so $E \sim \text{B}(9, 0.4)$, and find $P(E \geq 7)$

$\text{P}(E \geq 7) = 1 - \text{P}(E \leq 6) = 1 - 0.9750 = 0.0250 \text{ (4 d.p.)}$

5 a $\text{E}(X) = \frac{r}{0.4} = 15 \Rightarrow r = 15 \times 0.4 = 6$

b i $\text{P}(X = 10) = \binom{9}{5} \times (0.4)^6 \times (0.6)^4 = 0.0669 \text{ (4 d.p.)}$

5 b ii Let Y be the number of successes in 8 trials, so $Y \sim B(8, 0.4)$

$$\begin{aligned} P(X \leqslant 8) &= P(Y \geqslant 6) \\ &= 1 - P(Y \leqslant 5) \\ &= 1 - 0.9502 = 0.0498 \text{ (4 d.p.)} \end{aligned}$$

6 a $X \sim \text{Negative B}(r, p)$

$$E(X) = \frac{r}{p} = 6 \Rightarrow r = 6p$$

$$\text{Var}(X) = \frac{r(1-p)}{p^2} = 3 \Rightarrow r(1-p) = 3p^2$$

Substitute $r = 6p$ gives $6p(1-p) = 3p^2$

As $p \neq 0, 6(1-p) = 3p \Rightarrow p = \frac{6}{9} = \frac{2}{3}$

$r = 6p \Rightarrow r = 4$

b $P(X = 4) = \left(\frac{2}{3}\right)^4 = \frac{16}{81} = 0.1975 \text{ (4 d.p.)}$

7 a The attempts by each student are independent of each other and the probability of success remains the same for each student.

b $X \sim \text{Negative B}(5, 0.7)$

$$E(X) = \frac{5}{0.7} = \frac{50}{7} = 7.143 \text{ (3 d.p.)}$$

$$\text{Var}(X) = \frac{5(1-0.7)}{0.7^2} = \frac{1.5}{0.49} = 3.0612$$

$$\sigma = \sqrt{\text{Var}(X)} = 1.750 \text{ (3 d.p.)}$$

8 a $\left(\frac{1}{2}\right)^3 = \frac{1}{8} = 0.125$

Note the desired probability is not affected by the probability of the first coin landing on a particular face.
Another approach to this problem is to consider that there are 16 possible ways in which the coins can land ($2^4 = 16$), and only two combinations that show the same result: HHHH and TTTT. So the probability is $\frac{2}{16} = \frac{1}{8}$.

b $X \sim \text{Negative B}(3, 0.125)$

$$P(X = 6) = \binom{5}{2} \times (0.125)^3 \times (0.875)^3 = 0.0131 \text{ (4 d.p.)}$$

c $Y \sim \text{Negative B}(12, 0.125)$

$$E(Y) = \frac{12}{0.125} = 96$$

9 a $X \sim \text{Negative B}(3, p)$

$$E(X) = \frac{3}{p} = 18.75 \Rightarrow p = \frac{3}{18.75} = \frac{1}{6.25} = \frac{4}{25} = 0.16$$

9 b $\text{Var}(X) = \dfrac{3\left(1-\dfrac{4}{25}\right)}{\left(\dfrac{4}{25}\right)^2} = \dfrac{3\times21\times25\times25}{25\times16} = \dfrac{1575}{16} = 98.44$ (2 d.p.)

c $Y \sim \text{Negative B}(5, 0.24)$

$\text{E}(Y) = \dfrac{5}{0.24} = \dfrac{500}{24} = \dfrac{125}{6} = 20.83$ (2 d.p.)

d From part **c**, if Michelle takes 21 throws or more to hit treble twenty 5 times it would be more than expected. Let Z be the number of treble twenties she hits in 20 throws, $Z \sim \text{B}(20, 0.24)$

$\text{P}(Y > 20) = \text{P}(Z \leqslant 4) = 0.4561$ (4 d.p.)

10 a A negative binomial, $X \sim \text{Negative B}(r, p)$, where p is the probability of selecting a green marble.

b i $\text{E}(X) = \dfrac{r}{p} = 12 \Rightarrow r = 12p$

$\text{Var}(X) = \dfrac{r(1-p)}{p^2} = 6^2 \Rightarrow r(1-p) = 36p^2$

Substitute $r = 12p$ gives $12p(1-p) = 36p^2$

As $p \neq 0, (1-p) = 3p \Rightarrow p = \dfrac{1}{4} = 0.25$

There are 100p green marbles in the bag, so there are 25 green marbles $(100\times0.25 = 25)$

ii $r = 12p \Rightarrow r = 3$

c As each selected marble is not replaced, the probability of picking a green marble changes for the next selection. As the probability is not constant, the negative binomial model is not suitable.

d There are two ways of picking the second green marble on her third pick, picking in sequence 'green (G), not green (NG), green', or picking 'not green, green, green'. So the probability is:

$$\text{P}(G, NG, G) + \text{P}(NG, G, G) = \frac{25}{100}\times\frac{75}{99}\times\frac{24}{98} + \frac{75}{100}\times\frac{25}{99}\times\frac{24}{98} = \frac{50}{539} = 0.0928 \text{ (4 d.p.)}$$