Geometric and negative binomial distributions 3B

1 a $\mathrm{E}(X)=\frac{1}{0.2}=5$

b $\mathrm{Var}(X)=\frac{1-0.2}{0.2^2}=\frac{0.8}{0.04}=20$

2 a The probability of rolling a multiple of 3 is $\frac{1}{3}$

So $\mathrm{E}(X)=\frac{1}{\frac{1}{3}}=3$

b $\mathrm{Var}(X)=\frac{1-\frac{1}{3}}{\left(\frac{1}{3}\right)^2}=\frac{2\times 9}{3}=6$

3 Let X denote the number of attempts required to pass the driving test.

a $\mathrm{P}(X=3)=0.65(0.35)^2=0.0796$ (4 d.p.)

b $\mathrm{P}(X\geqslant 4)=0.35^3=0.0429$ (4 d.p.)

c i $\mathrm{E}(X)=\frac{1}{0.65}=\frac{20}{13}=1.5385$ (4 d.p.)

ii $\mathrm{Var}(X)=\frac{1-0.65}{0.65^2}=\frac{7}{20}\times\frac{400}{169}=\frac{140}{169}=0.8284$ (4 d.p.)

4 a $\mu=\frac{1}{p}\Rightarrow p=\frac{1}{\mu}\quad(\mu\neq 0)$, so in this case $p=\frac{1}{4}$

b $\mathrm{Var}(X)=\frac{1-\frac{1}{4}}{\left(\frac{1}{4}\right)^2}=\frac{3}{4}\times 16=12$

5 a $\mathrm{Var}(X)=20\Rightarrow\frac{1-p}{p^2}=20$

$20p^2+p-1=0$ Rearranging

$p=(5p-1)(4p+1)$ Solve by factorising

$p=\frac{-1\pm\sqrt{1^2-4(20)(-1)}}{2(20)}=\frac{-1\pm\sqrt{81}}{40}=\frac{-1\pm 9}{40}$ Or solve by using the quadratic equation

$\Rightarrow p=\frac{1}{5}=0.2\quad(\text{as } p>0)$

b $\mathrm{E}(X)=\frac{1}{0.2}=5$

6 a Let X denote the number of houses visited before receiving a donation.

$\text{Var}(X) = 380 \Rightarrow \dfrac{1-p}{p^2} = 380$

$380p^2 + p - 1 = 0$ Rearranging

$p = (20p-1)(19p+1)$ Solve by factorising

$p = \dfrac{-1 \pm \sqrt{1^2 - 4(380)(-1)}}{2(380)} = \dfrac{-1 \pm \sqrt{1521}}{760} = \dfrac{-1 \pm 39}{760}$ Or solve by using the quadratic equation

$\Rightarrow p = \dfrac{1}{20} = 0.05 \quad (\text{as } p > 0)$

b $\text{E}(X) = \dfrac{1}{0.05} = 20$

7 a Geometric, $X \sim \text{Geo}(p)$

b The probability of parking in the particular space is constant, i.e. the same on each attempt; and that each attempt is independent of any other.

c $\text{P}(X = 2) = p(1-p) = 0.16$

$p^2 - p + 0.16 = 0$ Rearranging

$p = \dfrac{1 \pm \sqrt{(-1)^2 - 4(0.16)}}{2} = \dfrac{1 \pm \sqrt{0.36}}{2} = \dfrac{1 \pm 0.6}{2}$ Solve by using the quadratic equation

$\Rightarrow p = 0.2 \text{ or } 0.8$

As $p < 0.5$, solution is $p = 0.2$

d $\text{E}(X) = \dfrac{1}{0.2} = 5$

e $\text{Var}(X) = \dfrac{1-0.2}{0.2^2} = \dfrac{0.8}{0.04} = 20$

8 Let X denote the number of attempts required to pull out a blue marble.

a i $\text{E}(X) = \dfrac{1}{0.15} = \dfrac{20}{3} = 6.67 \text{ (2 d.p.)}$

ii $\text{Var}(X) = \dfrac{1-0.15}{0.15^2} = \dfrac{17 \times 400}{20 \times 9} = \dfrac{340}{9} = 37.78 \text{ (2 d.p.)}$

b $\text{P}(X = 4) = 0.15(0.85)^3 = 0.0921 \text{ (4 d.p.)}$

c $\text{P}(X \geqslant 8) = 0.85^7 = 0.3206 \text{ (4 d.p.)}$

d $\text{P}\left(X < \dfrac{20}{3}\right) = \text{P}(X \leqslant 6) = 1 - 0.85^6 = 0.6229 \text{ (4 d.p.)}$

9 a The probability of the cat catching a fish is constant, i.e. the same on each attempt; and that each attempt is independent of any other.

b Let X denote the number of attempts required to catch a fish.

i $\text{P}(X = 2) = 0.12(0.88) = 0.1056$

ii $\text{P}(X \geqslant 3) = 0.88^2 = 0.7744$

9 c $\text{E}(X) = \dfrac{1}{0.12} = \dfrac{100}{12} = \dfrac{25}{3} = 8.33 \text{ (2 d.p.)}$

$\text{Var}(X) = \dfrac{1-0.12}{0.12^2} = \dfrac{0.88 \times 10000}{144} = \dfrac{8800}{144} = \dfrac{550}{9} = 61.11 \text{ (2 d.p.)}$

d $\text{P}(X=3) \times \text{P}(X=3) = \left(0.12 \times 0.88^2\right)^2 = 0.09293 \times 0.09293 = 0.0086 \text{ (4 d.p.)}$

e $\text{P}(X=2) \times \text{P}(X=3) = (0.12 \times 0.88)(0.12 \times 0.88^2) = 0.1056 \times 0.09293 = 0.0098 \text{ (4 d.p.)}$

10 a Assuming the faults occur independently and at random, and the long-term average faults per metre is constant, then use a Poisson distribution, $X \sim \text{Po}(0.8)$

b $\text{P}(X > 2) = 1 - \text{P}(X \leqslant 2)$

$= 1 - 0.9526 = 0.0474 \text{ (4 d.p.)}$

c Let Y denote the number of metre cloths cut before finding one with more than 2 faults. Then from part **b**, $Y \sim \text{Geo}(0.0474)$

$\text{P}(Y=7) = 0.0474(0.9256)^6 = 0.0354 \text{ (4 d.p.)}$

d $\text{E}(Y) = \dfrac{1}{0.0474} = 21 \text{ (to nearest whole number)}$

$\text{Var}(Y) = \dfrac{1-0.0474}{0.0474^2} = 424 \text{ (to nearest whole number)}$

e Find the probability of that there are 2 or more faults in a metre of cloth.

$\text{P}(X \geqslant 2) = 1 - \text{P}(X \leqslant 1) = 1 - 0.8088 = 0.1912$

Let Z denote the number of metre cloths cut before finding one with 2 of more faults. So $Z \sim \text{Geo}(0.1912)$. A roll is rejected if $Z \leqslant 2$

$\text{P}(Z \leqslant 2) = 1 - (1 - 0.1912)^2 = 0.3458$

The probability that two consecutive rolls are sent back is:

$\text{P}(Z \leqslant 2) \times \text{P}(Z \leqslant 2) = 0.3458^2 = 0.1196 \text{ (4 d.p.)}$