Probability distributions 2F

1 a $\mathrm{E}(X) = 12 \times 0.7 = 8.4$

b $\mathrm{Var}(X) = 12 \times 0.7 \times (1 - 0.7) = 2.52$

2 a $\mathrm{E}(X) = 0.4n = 3.2$

So $n = 8$

b $\mathrm{P}(X = 5) = \binom{8}{5}(0.4)^5(0.6)^3$

$= 56 \times 0.1024 \times 0.216 = 0.1239$ (4 d.p.)

c The answer can be found from the binomial cumulative distribution function tables on page 185 of the textbook or by using a calculator.

$\mathrm{P}(X \leqslant 2) = 0.3154$

3 $\mathrm{Var}(X) = 10p(1 - p) = 2.4$

So $p(1 - p) = 0.24$

$p^2 - p + 0.24 = 0$

$(p - 0.4)(p - 0.6) = 0$ factoring

$p = 0.4$ or 0.6

4 $\mathrm{Var}(X) = 15p(1 - p) = 2.4$

So $p(1 - p) = 0.16$

$p^2 - p + 0.16 = 0$

$(p - 0.2)(p - 0.8) = 0$ factoring

$p = 0.2$ or 0.8

5 $\mathrm{E}(X) = np = 4.8$ **(1)**

$\mathrm{Var}(X) = np(1 - p) = 2.88$ **(2)**

(2) ÷ **(1)** gives: $(1 - p) = \frac{2.88}{4.8} = 0.6 \Rightarrow p = 0.4$

From **(1)**: $np = 4.8 \Rightarrow n = \frac{4.8}{0.4} = 12$

6 a Let X be the number of heads obtained in 20 spins, so $X \sim \mathrm{B}(20, p)$

$\mathrm{Var}(X) = 20p(1 - p) = 4.2$

So $p(1 - p) = 0.21$

$p^2 - p + 0.21 = 0$

$(p - 0.3)(p - 0.7) = 0$ factoring

$p = 0.3$ (as $p < 0.5$)

b $\mathrm{P}(X = 7) = \binom{20}{7}(0.3)^7(0.7)^{13}$

$= 77\,520 \times 0.0002187 \times 0.0096889 = 0.1643$ (4 d.p.)

7 a Let Y be the number of times in 10 attempts that the canvasser gets a reply.
So $Y \sim B(10, 0.65)$

i $P(Y=5) = \binom{10}{5}(0.65)^5(0.35)^5$

$= 252 \times 0.11603 \times 0.005252 = 0.1536$ (4 d.p.)

ii $P(Y \geqslant 5) = 1 - P(Y \leqslant 4)$ find value by using a calculator

$= 1 - 0.0949 = 0.9051$

b Let X be the number of times in n attempts that the canvasser gets a reply.
So $X \sim B(n, 0.65)$

i Require that $E(X) = 0.65n = 78$
So $n = 120$

ii $\text{Var}(X) = np(p-1) = 120 \times 0.65 \times 0.35 = 27.3$

8 a Let X be the number of bars in 48 which are solid chocolate.
So $X \sim B(48, 0.04)$

$P(X \geqslant 2) = 1 - P(X \leqslant 1)$ find value by using a calculator

$= 1 - 0.4228 = 0.5772$

b Let Y be the number of boxes out of 120 which contain at least 2 solid chocolate bars.
So $Y \sim B(120, 0.5772)$

$E(X) = 120 \times 0.5772 = 69.264$

$\text{Var}(X) = 120 \times 0.5772 \times (1 - 0.5772)$

$= 120 \times 0.5772 \times 0.4228$

$= 29.28$ (2 d.p.)

9 a $X \sim B(5, p)$

$P(X \geqslant 1) = 1 - P(X = 0)$

$= 1 - \binom{5}{0} p^0 (1-p)^5 = 1 - (1-p)^5 = 0.83193$

So $(1-p)^5 = 0.16807$

$1 - p = 0.7 \Rightarrow p = 0.3$

b $E(X) = 5 \times 0.3 = 1.5$

$\text{Var}(X) = 5 \times 0.3 \times 0.7 = 1.05$

10 a Let X be the number of sixes in 5 throws of the dice.
From the data:

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{163 \times 0 + 208 \times 1 + 98 \times 2 + 28 \times 3 + 3 \times 4}{163208 + 98 + 28 + 3} = 1$$

$$\sigma^2 = \frac{\sum fx^2}{\sum f} - 1^2 = \frac{163 \times 0^2 + 208 \times 1^2 + 98 \times 2^2 + 28 \times 3^2 + 3 \times 4^2}{163208 + 98 + 28 + 3} = 1.8 - 1 = 0.8$$

b $X \sim B(5, p)$

$E(X) = 5p = 1$

So $p = 0.2$

10 c Using $X \sim B(5, 0.2)$, the expected frequencies can be found using the formula

$$\text{expected frequency for } x \text{ sixes} = 500 \times P(X = x) = 500 \times \binom{5}{x}(0.2)^x (0.8)^{5-x}$$

Number of sixes	0	1	2	3	4	5
Expected frequency	164	205	102	26	3	0

The data are very similar to the expected results from the proposed binomial model.

d Using the model, $\text{Var}(X) = 5 \times 0.2 \times 0.8 = 0.8$, which exactly matches the observed variance of the data. This further supports the suitability of the binomial model.

Challenge

a $E(X) = \sum x P(X - x)$

$$= \sum_{x=0}^{3} x \binom{3}{x} p^x (1-p)^{3-x}$$

$$= 0 + 1 \times 3p(1-p)^2 + 2 \times 3p^2(1-p) + 3 \times p^3$$

$$= 3p(1 - 2p + p^2) + 6p^2(1-p) + 3p^3$$

$$= 3p(1 - 2p + p^2 + 2p - 2p^2 + p^2) = 3p$$

b $\text{Var}(X) = \sum x^2 P(X - x) - (E(X))^2$

$$= \sum_{x=0}^{3} x^2 \binom{3}{x} p^x (1-p)^{3-x} - (3p)^2$$

$$= 0 + 1^2 \times 3p(1-p)^2 + 2^2 \times 3p^2(1-p) + 3^2 \times p^3 - 9p^2$$

$$= 3p(1 - 2p + p^2) + 12p^2(1-p) + 9p^3 - 9p^2$$

$$= 3p(1 - 2p + p^2 + 4p - 4p^2 + 3p^2 - 3p) = 3p(1-p)$$