Poisson distributions 2D

1 a \(X + Y \sim \text{Po}(3.3 + 2.7)\)
\(X + Y \sim \text{Po}(6)\)
\[
P(X + Y = 5) = \frac{e^{-6} \times 6^5}{5!} = 0.1606 \text{ (4 d.p.)}
\]
b Use the tables with \(\lambda = 6\)
\(P(X + Y \leq 7) = 0.7440\)
c Use the tables with \(\lambda = 6\)
\(P(X + Y > 4) = 1 - P(X + Y \leq 4) = 1 - 0.2851 = 0.7149\)

2 a \(A + B \sim \text{Po}(7.5)\)
\[
P(A + B = 7) = \frac{e^{-7.5} \times 7.5^7}{7!} = 0.1465 \text{ (4 d.p.)}
\]
b Use the tables with \(\lambda = 7.5\)
\(P(A + B \leq 5) = 0.2414\)
c Use the tables with \(\lambda = 7.5\)
\(P(A + B > 9) = 1 - P(A + B \leq 9) = 1 - 0.7764 = 0.2236\)

3 a As \(X\) and \(Y\) are independent
\[
P(X = 2 \text{ and } Y = 2) = P(X = 2) \times P(Y = 2)
= \frac{e^{-2.5} \times 2.5^2 \times e^{-3.5} \times 3.5^2}{2! \times 2!}
= 0.25652 \times 0.18496 = 0.0474 \text{ (4 d.p.)}
\]
b As \(X\) and \(Y\) are independent:
\[
P(X > 2 \text{ and } Y > 2) = (1 - P(X > 2)) \times (1 - P(Y > 2))
= (1 - 0.5438)(1 - 0.3208)
= 0.4562 \times 0.6792 = 0.3099
\]
c \(X + Y \sim \text{Po}(2.5 + 3.5)\), so \(X + Y \sim \text{Po}(6)\)
\[
P(X + Y = 5) = \frac{e^{-6} \times 6^5}{5!} = 0.1606 \text{ (4 d.p.)}
\]
d Use the tables with \(\lambda = 6\)
\(P(X + Y \leq 4) = 0.2851\)
4 \ a \ X \sim \text{Po}(3), \ Y \sim \text{Po}(5). \ As \ X \text{ and } Y \text{ are independent:}
\begin{align*}
P(X \geq 3 \text{ and } Y \geq 3) &= P(X \geq 3) \times P(Y \geq 3) \\
&= (1 - P(X \leq 2))(1 - P(Y \leq 2)) \\
&= (1 - 0.4232)(1 - 0.1247) \\
&= 0.5768 \times 0.8753 = 0.5049
\end{align*}

b \ X + Y \sim \text{Po}(8). \ Use \ the \ tables \ with \ \lambda = 8
\begin{align*}
P(X + Y \leq 6) &= 0.3134
\end{align*}

5 \ a \ Let \ X \text{ be the number of cars and } Y \text{ be the number of lorries passing in a 15-second period. So for a 15-second period } X \sim \text{Po}(6), \ Y \sim \text{Po}(2).
\begin{enumerate}
\item \ Assuming \ X \text{ and } Y \text{ are independent:}
\begin{align*}
P(X \geq 4 \text{ and } Y \geq 4) &= P(X \geq 4) \times P(Y \geq 4) \\
&= (1 - P(X \leq 3))(1 - P(Y \leq 3)) \\
&= (1 - 0.1512)(1 - 0.8571) \\
&= 0.8488 \times 0.1429 = 0.1213
\end{align*}
\item \ X + Y \sim \text{Po}(8). \ Use \ the \ tables \ with \ \lambda = 8
\begin{align*}
P(X + Y \leq 9) &= 0.7166
\end{align*}
\end{enumerate}

b \ It \ is \ assumed \ that \ the \ number \ of \ each \ vehicle \ type \ passing \ by \ follows \ a \ Poisson \ distribution \ (a \ constant \ mean \ rate \ over \ a \ set \ period \ does \ not \ necessarily \ imply \ a \ Poisson \ distribution \ within \ that \ period; \ for \ example, \ a \ set \ of \ traffic \ lights \ with \ a \ one \ minute \ cycle \ would \ allow \ a \ constant \ per-minute \ rate \ without \ the \ per-15-second \ rate \ being \ consistent). \ Another \ assumption \ is \ that \ the \ numbers \ of \ cars \ passing \ and \ trucks \ passing \ are \ independent \ (which \ would \ not \ be \ the \ case \ if \ there \ is \ traffic \ congestion, \ where \ the \ rates \ of \ each \ would \ be \ affected \ by \ a \ common \ external \ factor, \ and \ would \ in \ any \ case \ be \ unfeasible \ at \ large \ values \ where \ total \ road \ space \ would \ become \ a \ restriction, \ so \ that \ many \ lorries \ passing \ might \ preclude \ many \ cars \ also \ passing).

6 \ Let \ A \text{ and } B \text{ be the number of taxis ordered by companies A and B (respectively) on a given day. So } A \sim \text{Po}(1.25) \text{ and } B \sim \text{Po}(0.75), \text{ and } A \text{ and } B \text{ are independent.}
\begin{enumerate}
\item \ a \ P(A = 2) = \frac{e^{-1.25} \times 1.25^2}{2!} = 0.2238 \text{ (4 d.p.)}
\item \ b \ A + B \sim \text{Po}(2)
\begin{align*}
P(A + B = 2) &= \frac{e^{-2} \times 2^2}{2!} = 0.2707 \text{ (4 d.p.)}
\end{align*}
\item \ c \ Let \ C \text{ be the total number of taxis ordered by the two companies in a given 5-day week. } C \sim \text{Po}(10). \text{ Use \ the \ tables \ with } \lambda = 10
\begin{align*}
P(C < 10) &= P(C \leq 9) = 0.4579
\end{align*}
\end{enumerate}

7 \ Let \ C \text{ and } D \text{ be the number of times machines C and D (respectively) break down in a 12-week period. So } C \sim \text{Po}(1.2) \text{ and } D \sim \text{Po}(0.6), \text{ and } C \text{ and } D \text{ are independent.}
\begin{enumerate}
\item \ a \ As \ \lambda = 1.2, \text{ use \ a \ calculator \ to \ find \ the \ value.}
\begin{align*}
P(C \geq 1) &= 1 - P(C = 0) \\
&= 1 - \frac{e^{-1.2} \times 1.2^0}{0!} \\
&= 1 - 0.3012 = 0.6988 \text{ (4 d.p.)}
\end{align*}
\end{enumerate}
7 b \[ P(C \geq 1 \text{ and } D \geq 1) = P(C \geq 1) \times P(D \geq 1) = 0.6988 \left(1 - \frac{e^{-0.6} \times 0.6^0}{0!}\right) = 0.6988(1 - 0.5488) = 0.3153 \]

c \[ C + D \sim \text{Po}(1.8). \text{ Use a calculator to find the required value.} \]
\[ P(C + D = 3) = e^{-1.8} \times 1.8^3 \times 3! = 0.1607 \text{ (4 d.p.)} \]

8 a \[ \text{Let } A \text{ be the total number of calls received in a four-minute period. A rate of 3 calls in 5 minutes is equivalent to 2.4 calls in 4 minutes. So } A \sim \text{Po}(2.4). \]
\[ P(A = 3) = e^{-2.4} \times 2.4^3 ÷ 3! = 0.2090 \text{ (4 d.p.)} \]

b \[ \text{Let } B \text{ be the total number of calls received in a two-minute period. A rate of 3 calls in 5 minutes is equivalent to 1.2 calls in 2 minutes. So } B \sim \text{Po}(1.2). \]
\[ P(B \geq 2) = 1 - P(B \leq 1) = 1 - P(B = 0) - P(B = 1) = 1 - e^{-1.2} \times 1.2^0 ÷ 0! - e^{-1.2} \times 1.2^1 ÷ 1! = 1 - 0.30119 - 0.36143 = 0.3374 \text{ (4 d.p.)} \]

c \[ \text{Let } C \text{ be the total number of calls received in a ten-minute period. A rate of 3 calls in 5 minutes is equivalent to 6 calls in 10 minutes. So } C \sim \text{Po}(6). \text{ Use the tables with } \lambda = 6 \]
\[ P(C \leq 5) = 0.4457 \]
9 Let \( A, B \) and \( C \) be the number of times the ground, first-floor and second-floor photocopiers (respectively) break down in a given week. So \( A \sim \text{Po}(0.4), B \sim \text{Po}(0.2) \) and \( C \sim \text{Po}(0.8) \), and \( A, B \) and \( C \) are independent.

a As \( A, B \) and \( C \) are independent:
\[
P(A = 1 \text{ and } B = 1 \text{ and } C = 1) = P(A = 1) \times P(B = 1) \times P(C = 1)
\]
\[
= e^{-0.4} \times 0.4^1 \times 1! \times e^{-0.2} \times 0.2^1 \times 1! \times e^{-0.8} \times 0.8^1 \times 1!
\]
\[
= 0.26813 \times 0.16375 \times 0.35946
\]
\[
= 0.0158 \text{ (4 d.p.)}
\]

b \( A + B + C \sim \text{Po}(1.4) \). Use a calculator to find the required value.
\[
P(A + B + C \geq 1) = 1 - P(A + B + C = 0)
\]
\[
= 1 - e^{-1.4} \times 1.4^0 \times 0!
\]
\[
= 1 - 0.2466 = 0.7534 \text{ (4 d.p.)}
\]

c \( P(A + B + C = 2) = e^{-1.4} \times 1.4^2 \times 2! = 0.2417 \text{ (4 d.p.)} \)

10 a Let \( A, B \) and \( C \) be the number of personal, business and advertising emails (respectively) arriving in a given 30-minute period. Assume the number of each type of email arrives independently. So \( A \sim \text{Po}(0.9), B \sim \text{Po}(1.85) \) and \( C \sim \text{Po}(0.75) \), and \( A, B \) and \( C \) are independent.

\[
P(A \geq 1 \text{ and } B \geq 1 \text{ and } C \geq 1) = (1 - P(A = 0)) \times (1 - P(B = 0)) \times (1 - P(C = 0))
\]
\[
= (1 - e^{-0.9} \times 0.9^0 \times 0!) \times (1 - e^{-1.85} \times 1.85^0 \times 0!) \times (1 - e^{-0.75} \times 0.75^0 \times 0!)
\]
\[
= (1 - 0.4066) \times (1 - 0.1572) \times (1 - 0.4724)
\]
\[
= 0.5934 \times 0.8428 \times 0.5276
\]
\[
= 0.2639 \text{ (4 d.p.)}
\]

b Let \( D \) be the total number of emails in an 8-hour period.
Total hourly email rate: \( 1.8 + 3.7 + 1.5 = 7 \), so \( D \sim \text{Po}(56) \).
\[
P(D > 50) = 1 - P(D \leq 50) = 1 - 0.2343 = 0.7657 \text{ (4 d.p.)}
\]

c Let \( X \) be the number of days out of a 5-day working week on which the director receives more than 50 emails. As the probability of receiving more than 50 emails is \( P(D > 50) = 0.7657 \) (from part b), the model is \( X \sim \text{B}(5, 0.7657) \).
\[
P(X = 2) = \binom{5}{2} (0.7657)^2 (1 - 0.7657)^3 = 0.0754 \text{ (4 d.p.)}
\]
**Challenge**

a If \( Q = 0 \) then because \( X \) and \( Y \) cannot take negative values, \( X = Y = 0 \).

\[
P(Q = 0) = P(X = 0 \text{ and } Y = 0) = P(X = 0) \times P(Y = 0) \quad \text{(independence)}
\]
\[
= e^{-\lambda} \times e^{-\mu} = e^{-(\lambda+\mu)}
\]

Alternatively, by addition of Poisson distributions \( X + Y \sim Po(\lambda + \mu) \)

So \( P(Q = 0) = P(X + Y = 0) = \frac{e^{-(\lambda+\mu)} \times (\lambda + \mu)^0}{0!} \)

Which, \( (\lambda + \mu)^0 = 1 \) and \( 0! = 1 \), gives

\( P(Q = 0) = e^{-(\lambda+\mu)} \)

b If \( Q = 1 \) then \((X,Y) = (0,1) \) or \((1,0) \)

\[
P(Q = 1) = P\left(\left( X = 0 \text{ and } Y = 1 \right) \text{ or } \left( X = 1 \text{ and } Y = 0 \right)\right)
\]
\[
= P(X = 0 \text{ and } Y = 1) + P(X = 1 \text{ and } Y = 0) \quad \text{(mutually exclusive)}
\]
\[
= P(X = 0) \times P(Y = 1) + P(X = 1) \times P(Y = 0) \quad \text{(independent)}
\]
\[
= \left( e^{-\lambda} \times \frac{e^{-\mu}}{1!} \right) + \left( e^{-\lambda} \times \frac{\lambda}{1!} \times e^{-\mu} \right)
\]
\[
= (\lambda + \mu)e^{-(\lambda+\mu)}
\]

Alternatively, as \( X + Y \sim Po(\lambda + \mu) \)

So \( P(Q = 1) = P(X + Y = 1) = \frac{e^{-(\lambda+\mu)} \times (\lambda + \mu)^1}{1!} \)

Which, \( (\lambda + \mu)^1 = \lambda + \mu \) and \( 1! = 1 \), gives

\( P(Q = 1) = (\lambda + \mu)e^{-(\lambda+\mu)} \)