Poisson distributions 2D

1 a $X+Y \sim \text{Po}(3.3+2.7)$

$X+Y \sim \text{Po}(6)$

$$\text{P}(X+Y=5)=\frac{e^{-6}\times 6^5}{5!}=0.1606 \text{ (4 d.p.)}$$

b Use the tables with $\lambda=6$

$\text{P}(X+Y \leqslant 7)=0.7440$

c Use the tables with $\lambda=6$

$\text{P}(X+Y>4)=1-\text{P}(X+Y \leqslant 4)=1-0.2851=0.7149$

2 a $A+B \sim \text{Po}(7.5)$

$$\text{P}(A+B=7)=\frac{e^{-7.5}\times 7.5^7}{7!}=0.1465 \text{ (4 d.p.)}$$

b Use the tables with $\lambda=7.5$

$\text{P}(A+B \leqslant 5)=0.2414$

c Use the tables with $\lambda=7.5$

$\text{P}(A+B>9)=1-\text{P}(A+B \leqslant 9)=1-0.7764=0.2236$

3 a As X and Y are independent

$$\begin{aligned}\text{P}(X=2 \text{ and } Y=2)&=\text{P}(X=2)\times\text{P}(Y=2)\\&=\frac{e^{-2.5}\times 2.5^2}{2!}\times\frac{e^{-3.5}\times 3.5^2}{2!}\\&=0.25652\times 0.18496=0.0474 \text{ (4 d.p.)}\end{aligned}$$

b As X and Y are independent:

$$\begin{aligned}\text{P}(X>2 \text{ and } Y>2)&=\text{P}(X>2)\times\text{P}(Y>2)\\&=\big(1-\text{P}(X>2)\big)\big(1-\text{P}(Y>2)\big)\\&=(1-0.5438)(1-0.3208)\\&=0.4562\times 0.6792=0.3099\end{aligned}$$

c $X+Y \sim \text{Po}(2.5+3.5)$, so $X+Y \sim \text{Po}(6)$

$$\text{P}(X+Y=5)=\frac{e^{-6}\times 6^5}{5!}=0.1606 \text{ (4 d.p.)}$$

d Use the tables with $\lambda=6$

$\text{P}(X+Y \leqslant 4)=0.2851$

4 a $X \sim \text{Po}(3),\ Y \sim \text{Po}(5)$. As X and Y are independent:

$$\begin{aligned}\text{P}(X \geqslant 3 \text{ and } Y \geqslant 3) &= \text{P}(X \geqslant 3)\times\text{P}(Y \geqslant 3)\\ &= \left(1-\text{P}(X \leqslant 2)\right)\left(1-\text{P}(Y \leqslant 2)\right)\\ &= (1-0.4232)(1-0.1247)\\ &= 0.5768\times 0.8753 = 0.5049\end{aligned}$$

b $X+Y \sim \text{Po}(8)$. Use the tables with $\lambda = 8$

$\text{P}(X+Y \leqslant 6) = 0.3134$

5 a Let X be the number of cars and Y be the number of lorries passing in a 15-second period. So for a 15-second period $X \sim \text{Po}(6),\ Y \sim \text{Po}(2)$.

i Assuming X and Y are independent:

$$\begin{aligned}\text{P}(X \geqslant 4 \text{ and } Y \geqslant 4) &= \text{P}(X \geqslant 4)\times\text{P}(Y \geqslant 4)\\ &= \left(1-\text{P}(X \leqslant 3)\right)\left(1-\text{P}(Y \leqslant 3)\right)\\ &= (1-0.1512)(1-0.8571)\\ &= 0.8488\times 0.1429 = 0.1213\end{aligned}$$

ii $X+Y \sim \text{Po}(8)$. Use the tables with $\lambda = 8$

$\text{P}(X+Y \leqslant 9) = 0.7166$

b It is assumed that the number of each vehicle type passing by follows a Poisson distribution (a constant mean rate over a set period does not necessarily imply a Poisson distribution within that period; for example, a set of traffic lights with a one minute cycle would allow a constant per-minute rate without the per-15-second rate being consistent).
Another assumption is that the numbers of cars passing and trucks passing are independent (which would not be the case if there is traffic congestion, where the rates of each would be affected by a common external factor, and would in any case be unfeasible at large values where total road space would become a restriction, so that many lorries passing might preclude many cars also passing).

6 Let A and B be the number of taxis ordered by companies A and B (respectively) on a given day. So $A \sim \text{Po}(1.25)$ and $B \sim \text{Po}(0.75)$, and A and B are independent.

a $\text{P}(A=2) = \dfrac{e^{-1.25}\times 1.25^2}{2!} = 0.2238 \text{ (4 d.p.)}$

b $A+B \sim \text{Po}(2)$

$\text{P}(A+B=2) = \dfrac{e^{-2}\times 2^2}{2!} = 0.2707 \text{ (4 d.p.)}$

c Let C be the total number of taxis ordered by the two companies in a given 5-day week.
$C \sim \text{Po}(10)$. Use the tables with $\lambda = 10$

$\text{P}(C<10) = \text{P}(C \leqslant 9) = 0.4579$

7 Let C and D be the number of times machines C and D (respectively) break down in a 12-week period. So $C \sim \text{Po}(1.2)$ and $D \sim \text{Po}(0.6)$, and C and D are independent.

a As $\lambda = 1.2$, use a calculator to find the value.

$$\begin{aligned}\text{P}(C \geqslant 1) &= 1-\text{P}(C=0)\\ &= 1-\frac{e^{-1.2}\times 1.2^0}{0!}\\ &= 1-0.3012 = 0.6988 \text{ (4 d.p.)}\end{aligned}$$

7 b $P(C \geqslant 1 \text{ and } D \geqslant 1) = P(C \geqslant 1) \times P(D \geqslant 1)$

$$= 0.6988\left(1 - \frac{e^{-0.6} \times 0.6^0}{0!}\right)$$

$$= 0.6988(1 - 0.5488)$$

$$= 0.6988 \times 0.4512 = 0.3153$$

c $C + D \sim \text{Po}(1.8)$. Use a calculator to find the required value.

$$P(C + D = 3) = \frac{e^{-1.8} \times 1.8^3}{3!} = 0.1607 \text{ (4 d.p.)}$$

8 a Let A be the total number of calls received in a four-minute period. A rate of 3 calls in 5 minutes is equivalent to 2.4 calls in 4 minutes. So $A \sim \text{Po}(2.4)$.

$$P(A = 3) = \frac{e^{-2.4} \times 2.4^3}{3!} = 0.2090 \text{ (4 d.p.)}$$

b Let B be the total number of calls received in a two-minute period. A rate of 3 calls in 5 minutes is equivalent to 1.2 calls in 2 minutes. So $B \sim \text{Po}(1.2)$.

$$P(B \geqslant 2) = 1 - P(B \leqslant 1) = 1 - P(B = 0) - P(B = 1)$$

$$= 1 - \frac{e^{-1.2} \times 1.2^0}{0!} - \frac{e^{-1.2} \times 1.2^1}{1!}$$

$$= 1 - 0.30119 - 0.36143 = 0.3374 \text{ (4 d.p.)}$$

c Let C be the total number of calls received in a ten-minute period. A rate of 3 calls in 5 minutes is equivalent to 6 calls in 10 minutes. So $C \sim \text{Po}(6)$. Use the tables with $\lambda = 6$

$P(C \leqslant 5) = 0.4457$

9 Let A, B and C be the number of times the ground, first-floor and second-floor photocopiers (respectively) break down in a given week. So $A \sim \text{Po}(0.4)$, $B \sim \text{Po}(0.2)$ and $C \sim \text{Po}(0.8)$, and A, B and C are independent.

a As A, B and C are independent:

$$\begin{aligned}\text{P}(A=1 \text{ and } B=1 \text{ and } C=1) &= \text{P}(A=1)\times\text{P}(B=1)\times\text{P}(C=1)\\ &= \frac{e^{-0.4}\times 0.4^1}{1!}\times\frac{e^{-0.2}\times 0.2^1}{1!}\times\frac{e^{-0.8}\times 0.8^1}{1!}\\ &= 0.26813\times 0.16375\times 0.35946\\ &= 0.0158 \text{ (4 d.p.)}\end{aligned}$$

b $A+B+C \sim \text{Po}(1.4)$. Use a calculator to find the required value.

$$\begin{aligned}\text{P}(A+B+C \geqslant 1) &= 1-\text{P}(A+B+C=0)\\ &= 1-\frac{e^{-1.4}\times 1.4^0}{0!}\\ &= 1-0.2466 = 0.7534 \text{ (4 d.p.)}\end{aligned}$$

c $\text{P}(A+B+C=2) = \dfrac{e^{-1.4}\times 1.4^2}{2!} = 0.2417 \text{ (4 d.p.)}$

10 a Let A, B and C be the number of personal, business and advertising emails (respectively) arriving in a given 30-minute period. Assume the number of each type of email arrives independently. So $A \sim \text{Po}(0.9)$, $B \sim \text{Po}(1.85)$ and $C \sim \text{Po}(0.75)$, and A, B and C are independent.

$$\begin{aligned}\text{P}(A \geqslant 1 \text{ and } B \geqslant 1 \text{ and } C \geqslant 1) &= (1-\text{P}(A=0))\times(1-\text{P}(B=0))\times(1-\text{P}(C=0))\\ &= \left(1-\frac{e^{-0.9}\times 0.9^0}{0!}\right)\times\left(1-\frac{e^{-1.85}\times 1.85^0}{0!}\right)\times\left(1-\frac{e^{-0.75}\times 0.75^0}{0!}\right)\\ &= (1-0.4066)\times(1-0.1572)\times(1-0.4724)\\ &= 0.5934\times 0.8428\times 0.5276\\ &= 0.2639 \text{ (4 d.p.)}\end{aligned}$$

b Let D be the total number of emails in an 8-hour period.
Total hourly email rate: 1.8 + 3.7 + 1.5 = 7, so $D \sim \text{Po}(56)$.

$\text{P}(D > 50) = 1-\text{P}(D \leqslant 50) = 1-0.2343 = 0.7657 \text{ (4 d.p.)}$

c Let X be the number of days out of a 5-day working week on which the director receives more than 50 emails. As the probability of receiving more than 50 emails is $\text{P}(D>50) = 0.7657$ (from part **b**), the model is $X \sim \text{B}(5, 0.7657)$.

$$\text{P}(X=2) = \binom{5}{2}(0.7657)^2(1-0.7657)^3 = 0.0754 \text{ (4 d.p.)}$$

Challenge

a If $Q = 0$ then because X and Y cannot take negative values, $X = Y = 0$.

$$\begin{aligned}P(Q=0) &= P(X=0 \text{ and } Y=0)\\ &= P(X=0)\times P(Y=0) \quad \text{(independence)}\\ &= e^{-\lambda}\times e^{-\mu}\\ &= e^{-(\lambda+\mu)}\end{aligned}$$

Alternatively, by addition of Poisson distributions $X + Y \sim \text{Po}(\lambda+\mu)$

So $P(Q=0) = P(X+Y=0) = \dfrac{e^{-(\lambda+\mu)}\times(\lambda+\mu)^0}{0!}$

Which, $(\lambda+\mu)^0 = 1$ and $0! = 1$, gives

$P(Q=0) = e^{-(\lambda+\mu)}$

b If $Q = 1$ then $(X, Y) = (0,1)$ or $(1,0)$

$$\begin{aligned}P(Q=1) &= P\big((X=0 \text{ and } Y=1) \text{ or } (X=1 \text{ and } Y=0)\big)\\ &= P(X=0 \text{ and } Y=1) + P(X=1 \text{ and } Y=0) \quad \text{(mutually exclusive)}\\ &= P(X=0)\times P(Y=1) + P(X=1)\times P(Y=0) \quad \text{(independent)}\\ &= \left(e^{-\lambda}\times\frac{e^{-\mu}\mu^1}{1!}\right)+\left(\frac{e^{-\lambda}\lambda^1}{1!}\times e^{-\mu}\right)\\ &= (\lambda+\mu)e^{-(\lambda+\mu)}\end{aligned}$$

Alternatively, as $X + Y \sim \text{Po}(\lambda+\mu)$

So $P(Q=1) = P(X+Y=1) = \dfrac{e^{-(\lambda+\mu)}\times(\lambda+\mu)^1}{1!}$

Which, $(\lambda+\mu)^1 = \lambda+\mu$ and $1! = 1$, gives

$P(Q=1) = (\lambda+\mu)e^{-(\lambda+\mu)}$