Discrete random variables Mixed exercise 1

1 a The probability distribution for X is:

x	1	2	3	4	5	6
$\mathbf{P(X=x)}$	$\frac{1}{21}$	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{6}{21}$

b $P(2 < X \le 5) = P(X=3) + P(X=4) + P(X=5) = \frac{3}{21} + \frac{4}{21} + \frac{5}{21} = \frac{12}{21} = 7$

c $E(X) = 1 \times \frac{1}{21} + 2 \times \frac{2}{21} + 3 \times \frac{3}{21} + 4 \times \frac{4}{21} + 5 \times \frac{5}{21} + 6 \times \frac{6}{21}$

$= \frac{1}{21}(1+4+9+16+25+36) = \frac{91}{21} = \frac{13}{3}$

d $E(X^2) = 1 \times \frac{1}{21} + 4 \times \frac{2}{21} + 9 \times \frac{3}{21} + 16 \times \frac{4}{21} + 25 \times \frac{5}{21} + 36 \times \frac{6}{21}$

$= \frac{1}{21}(1+8+27+64+125+216) = \frac{441}{21} = 21$

$\text{Var}(X) = E(X^2) - (E(X))^2$

$= 21 - \left(\frac{13}{3}\right)^2 = 21 - \frac{169}{9}$

$= \frac{189}{9} - \frac{169}{9} = \frac{20}{9} = 2.22$ (2 d.p.)

e $\text{Var}(3-2X) = \text{Var}(-2X+3)$

$= (-2)^2 \text{Var}(X)$

$= 4 \times \frac{20}{9} = \frac{80}{9} = 8.89$ (2 d.p.)

f $E(X^3) = \sum x^3 P(X=x)$

$= 1^3 \times \frac{1}{21} + 2^3 \times \frac{2}{21} + 3^3 \times \frac{3}{21} + 4^3 \times \frac{4}{21} + 5^3 \times \frac{5}{21} + 6^3 \times \frac{6}{21}$

$= \frac{1}{21}(1+16+81+256+625+1296)$

$= \frac{2275}{21} = \frac{325}{3} = 108.33$ (2 d.p.)

2 a Probabilities sum to 1, so:

$0.1 + 0.2 + 0.3 + r + 0.1 + 0.1 = 1$

$r = 1 - 0.8 = 0.2$

b $P(-1 \le X < 2) = P(X=-1) + P(X=0) + P(X=1) = 0.2 + 0.3 + 0.2 = 0.7$

c $E(X) = -2 \times 0.1 + (-1) \times 0.1 + 0 \times 0.3 + 1 \times 0.2 + 2 \times 0.1 + 3 \times 0.1$

$= -0.2 - 0.2 + 0.2 + 0.2 + 0.3 = 0.3$

$E(2X+3) = 2E(X) + 3 = (2 \times 0.3) + 3 = 3.6$

2 d $E(X^2) = 4\times0.1 + 1\times0.1 + 0\times0.3 + 1\times0.2 + 4\times0.1 + 9\times0.1$
$= 0.4 + 0.2 + 0.2 + 0.4 + 0.9 = 2.1$

$\text{Var}(X) = E(X^2) - (E(X))^2$
$= 2.1 - (0.3)^2 = 2.1 - 0.09 = 2.01$

$\text{Var}(2X+3) = 2^2\,\text{Var}(X)$
$= 4\times2.01 = 8.04$

3 a Probabilities sum to 1, so:

$\frac{1}{5} + b + b + \frac{1}{5} = 1$

$2b = 1 - \frac{2}{5} = \frac{3}{5}$

$b = \frac{3}{10}$

b $E(X) = 0\times\frac{1}{5} + 1\times\frac{3}{10} + 2\times\frac{5}{10} = \frac{13}{10} = 1.3$

c $E(X^2) = 0\times\frac{1}{5} + 1\times\frac{3}{10} + 4\times\frac{5}{10} = \frac{23}{10} = 2.3$

$\text{Var}(X) = E(X^2) - (E(X))^2$
$= 2.3 - 1.3^2 = 2.3 - 1.69 = 0.61$

d $P(X \le 1.5) = P(X=0) + P(X=1) = \frac{1}{5} + \frac{3}{10} = 0.5$

4 a Probabilities sum to 1, so:

$k(1-0) + k(1-1) + k(2-1) + k(3-1) = 1$

$k + k + 2k = 1$

$4k = 1$

$k = \frac{1}{4} = 0.25$

b The probability distribution for X is:

x	0	1	2	3
$P(X=x)$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$

$E(X) = 0\times\frac{1}{4} + 1\times0 + 2\times\frac{1}{4} + 3\times\frac{1}{2} = \frac{1}{2} + \frac{3}{2} = 2$

$E(X^2) = 0\times\frac{1}{4} + 1\times0 + 4\times\frac{1}{4} + 9\times\frac{1}{2} = \frac{11}{2} = 5.5$

c $\text{Var}(X) = E(X^2) - (E(X))^2 = 5.5 - 2^2 = 5.5 - 4 = 1.5$

$\text{Var}(2X-2) = 2^2\,\text{Var}(X) = 4\times1.5 = 6$

5 a $P(1 < X \leqslant 2) = P(X = 2) = \frac{1}{8}$

b $E(X) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8} = \frac{1}{2} + \frac{1}{4} + \frac{3}{8} = \frac{9}{8}$

c $E(3X - 1) = 3E(X) - 1 = \frac{27}{8} - 1 = \frac{19}{8}$

d $E(X^2) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 4 \times \frac{1}{8} + 9 \times \frac{1}{8} = \frac{1}{2} + \frac{1}{2} + \frac{9}{8} = \frac{17}{8}$

$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{17}{8} - \left(\frac{9}{8}\right)^2 = \frac{136}{64} - \frac{81}{64} = \frac{55}{64}$

e $E(\log(X + 1)) = \sum \log(X + 1) P(X = x)$

$= \log(0 + 1) \times \frac{1}{4} + \log(1 + 1) \times \frac{1}{2} + \log(2 + 1) \times \frac{1}{8} + \log(3 + 1) \times \frac{1}{8}$

$= 0 + \frac{1}{2}\log 2 + \frac{1}{8}\log 3 + \frac{1}{8}\log 4$

$= 0.5 \times 0.30102 + 0.125 \times 0.47712 + 0.125 \times 0.60206$

$= 0.2854$ (4 d.p.)

6 a The probability distribution for X^2 is:

x^2	1	4	9	19
$P(X^2 = x^2)$	0.4	0.2	0.1	0.3

$P(3 < X^2 < 10) = 0.2 + 0.1 = 0.3$

b $E(X) = 1 \times 0.4 + 2 \times 0.2 + 3 \times 0.1 + 4 \times 0.3 = 2.3$

c $E(X^2) = 1 \times 0.4 + 4 \times 0.2 + 9 \times 0.1 + 16 \times 0.3 = 6.9$

$\text{Var}(X) = E(X^2) - (E(X))^2$

$= 6.9 - (2.3)^2 = 6.9 - 5.29 = 1.61$

d $E\left(\frac{3 - X}{2}\right) = \frac{3}{2} - \frac{1}{2}E(X) = \frac{3}{2} - \frac{2.3}{2} = \frac{0.7}{2} = 0.35$

e $E\left(\sqrt{X}\right) = \sum \sqrt{X}\, P(X = x)$

$= 1 \times 0.4 + \sqrt{2} \times 0.2 + \sqrt{3} \times 0.1 + \sqrt{4} \times 0.3$

$= 0.4 + 0.2828 + 0.1732 + 0.6$

$= 1.4560$ (4 d.p.)

f $E\left(2^{-X}\right) = \sum 2^{-X}\, P(X = x)$

$= 2^{-1} \times 0.4 + 2^{-2} \times 0.2 + 2^{-3} \times 0.1 + 2^{-4} \times 0.3$

$= 0.5 \times 0.4 + 0.25 \times 0.2 + 0.125 \times 0.1 + 0.0625 \times 0.3$

$= 0.2 + 0.05 + 0.0125 + 0.01875$

$= 0.28125$

7 a A discrete uniform distribution.

b Any distribution where all the probabilities are the same. An example is throwing a fair die.

c There are 5 possible values. So as the variable has discrete uniform distribution, each value has a $\frac{1}{5}$ (= 0.2) probability. E(X) can be found by symmetry, as the probability distribution is uniform, or by:

$\text{E}(X) = 0.2(0+1+2+3+4) = 0.2 \times 10 = 2$

d $\text{E}(X^2) = 0.2(0+1+4+9+16) = 0.2 \times 30 = 6$

$\text{Var}(X) = E(X^2) - (E(X))^2 = 6 - 2^2 = 6 - 4 = 2$

8 a Probabilities sum to 1, so:

$0.1 + p + q + 0.3 + 0.1 = 1$

$p + q = 0.5$ **(1)**

$\text{E}(X) = 0.1 + 2p + 3q + 1.2 + 0.5 = 3.1$

$2p + 3q = 1.3$ **(2)**

b Multiply equation (**1**) by 2:

$2p + 2q = 1$ **(3)**

Subtract equation (**3**) from equation (**2**)

$q = 0.3$

Substitute for q in equation (**1**)

$p + 0.3 = 0.5 \Rightarrow p = 0.2$

c $\text{E}(X) = 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.3 + 4 \times 0.3 + 5 \times 0.1$

$= 0.1 + 0.4 + 0.9 + 1.2 + 0.5 = 3.1$

$\text{E}(X^2) = 1 \times 0.1 + 4 \times 0.2 + 9 \times 0.3 + 16 \times 0.3 + 25 \times 0.1$

$= 0.1 + 0.8 + 2.7 + 4.8 + 2.5 = 10.9$

$\text{Var}(X) = E(X^2) - (E(X))^2 = 10.9 - (3.1)^2 = 10.9 - 9.61 = 1.29$

d $\text{Var}(2X+3) = 2^2 \text{Var}(X) = 4 \times 1.29 = 5.16$

9 a The probability distribution for X is:

x	1	2	3	4	5
P($X = x$)	k	2k	k	2k	3k

Probabilities sum to 1, so:

$k + 2k + k + 2k + 3k = 9k = 1$

$k = \frac{1}{9}$

b $\text{E}(X) = 1k + 2 \times 2k + 3k + 4 \times 2k + 5 \times 3k$

$= 31k = \frac{31}{9}$

9 c $E(X^2) = 1k + 4\times 2k + 9k + 16\times 2k + 25\times 3k$

$= 125k = \frac{125}{9}$

$\text{Var}(X) = E(X^2) - (E(X))^2$

$= \frac{125}{9} - \left(\frac{31}{9}\right)^2 = \frac{1125}{81} - \frac{961}{81} = \frac{164}{81}$

$= 2.02$ (3 s.f.)

d $\text{Var}(3-2X) = (-2)^2\,\text{Var}(X) = 4\times 2.02 = 8.1$ (1 d.p.)

10 a $E(X) = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2} = 3.5$

$E(X^2) = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}$

$\text{Var}(X) = E(X^2) - (E(X))^2$

$= \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{546}{36} - \frac{441}{36} = \frac{105}{36} = \frac{35}{12}$

b $E(2X-1) = 2E(X) - 1 = 7 - 1 = 6$

c $\text{Var}(3-2X) = (-2)^2\,\text{Var}(X) = 4\times\frac{35}{12} = \frac{35}{3} = 11.67$ (2 d.p.)

d $E\left(2^X\right) = \sum 2^X\,P(X = x)$

$= \frac{1}{6}(2+4+8+16+32+64) = \frac{126}{6} = 21$

11 a The probability distribution for X is:

x	1	2	3	4
$P(X = x)$	$\frac{2}{26}$	$\frac{5}{26}$	$\frac{8}{26}$	$\frac{11}{26}$

b $P(2 < X \leqslant 4) = P(X = 3) + P(X = 4) = \frac{19}{26} = 0.731$ (3 d.p.)

c $E(X) = 1\times\frac{2}{26} + 2\times\frac{5}{26} + 3\times\frac{8}{26} + 4\times\frac{11}{26}$

$= \frac{2+10+24+44}{26} = \frac{80}{26} = \frac{40}{13}$

d $E(X^2) = 1\times\frac{2}{26} + 4\times\frac{5}{26} + 9\times\frac{8}{26} + 16\times\frac{11}{26}$

$= \frac{2+20+72+176}{26} = \frac{270}{26} = \frac{135}{13}$

$\text{Var}(X) = E(X^2) - (E(X))^2$

$= \frac{135}{13} - \left(\frac{40}{13}\right)^2 = \frac{1755}{169} - \frac{1600}{169} = \frac{155}{169} = 0.92$ (2 s.f.)

11 e $\text{Var}(1-3X) = (-3)^2\,\text{Var}(X) = 9\times\frac{155}{169} = \frac{1395}{169} = 8.3\ (2\text{ s.f.})$

12 a $\text{E}(3Y+1) = 3\text{E}(Y)+1 = 3\times 2+1 = 7$

b $\text{E}(2-3Y) = 2-3\text{E}(Y) = 2-3\times 2 = -4$

c $\text{Var}(3Y+1) = 3^2\,\text{Var}(Y) = 9\times 9 = 81$

d $\text{Var}(2-3Y) = (-3)^2\,\text{Var}(Y) = 9\times 9 = 81$

e $\text{E}(Y^2) = \text{Var}(Y)+(E(Y))^2 = 9+2^2 = 13$

f $\text{E}((Y-1)(Y+1)) = \text{E}(Y^2-1) = \text{E}(Y^2)-1 = 13-1 = 12$

13 As the random variable T has a standard deviation of 5, $\text{Var}(T) = 5^2 = 25$

$\text{E}(S) = \text{E}(3T+4) = 3\text{E}(T)+4 = 3\times 20+4 = 64$

$\text{Var}(S) = \text{Var}(3T+4) = 3^2\,\text{Var}(T) = 9\times 25 = 225$

14 a The probability distribution for X is:

x	1	2	3
P(*X* = *x*)	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{3}{8}$

b $\text{E}(X) = 1\times\frac{1}{4}+2\times\frac{3}{8}+3\times\frac{3}{8} = \frac{17}{8} = 2.125$

c $\text{E}(X^2) = 1\times\frac{1}{4}+4\times\frac{3}{8}+9\times\frac{3}{8} = \frac{41}{8} = 5.125$

$\text{Var}(X) = E(X^2)-(E(X))^2$

$= \frac{41}{8}-\left(\frac{17}{8}\right)^2 = \frac{328}{64}-\frac{289}{64} = \frac{39}{64} = 0.609\ (3\text{ d.p.})$

d $\text{E}(2X+1) = 2\text{E}(X)+1 = \frac{17}{4}+1 = \frac{21}{4} = 5.25$

e $\text{Var}(3X-1) = (3)^2\,\text{Var}(X) = 9\times\frac{39}{64} = \frac{351}{64} = 5.484\ (3\text{ d.p.})$

15 a $\text{E}(X) = -0.2+0+0.2+0.2 = 0.2$

b $\text{E}(X^2) = 0.2+0+0.2+0.4 = 0.8$

$\text{Var}(X) = 0.8-0.2^2 = 0.8-0.04 = 0.76$

c $\text{E}\left(\frac{1}{3}X+1\right) = \frac{1}{3}\text{E}(X)+1 = \frac{1}{3}\times 0.2+1 = \frac{1}{3}\times\frac{1}{5}+1 = \frac{16}{15} = 1.067\ (3\text{ d.p.})$

d $\text{Var}\left(\frac{1}{3}X+1\right) = \left(\frac{1}{3}\right)^2\text{Var}(X) = \frac{1}{9}\times 0.76 = \frac{1}{9}\times\frac{19}{25} = \frac{19}{225} = 0.0844\ (3\text{ d.p.})$

16 a Probabilities sum to 1, so:

$0.1+0.3+a+b=1$

$a+b=0.6$ **(1)**

Rearrange the equation for Y to get X in terms of Y:

$3X=Y+1\Rightarrow X=\frac{1}{3}Y+\frac{1}{3}$

$E(X)=E\left(\frac{1}{3}Y+\frac{1}{3}\right)=\frac{1}{3}E(Y)+\frac{1}{3}=\frac{1}{3}\times\frac{11}{10}+\frac{1}{3}=\frac{21}{30}=0.7$

$\mathrm{E}(X)=\sum x\,\mathrm{P}(X=x)$

$=-0.1+a+2b=0.7$

So $a+2b=0.8$ **(2)**

Subtract equation (**1**) from equation (**2**), which gives:

$b=0.2$

So by substituting the value for b in equation (**1**)

$a+0.2=0.6\Rightarrow a=0.4$

b $\mathrm{E}(X^2)=1\times0.1+0\times0.3+1\times0.4+4\times0.2=1.3$

$\mathrm{Var}(X)=E(X^2)-(E(X))^2=1.3-0.7^2=1.3-0.49=0.81$

c $\mathrm{Var}(Y)=\mathrm{Var}(1-3X)=(-3)^2\,\mathrm{Var}(X)=9\times0.81=7.29$

d $\mathrm{P}(Y+2>X)=\mathrm{P}(3X-1+2>X)$

$=P(2X>-1)=P(X>-0.5)$

$=0.3+0.4+0.2=0.9$

17 a Probabilities sum to 1, so:

$2a+2b+c=1$ **(1)**

Rearrange the equation for Y to get X in terms of Y:

$Y=\frac{2-3X}{5}\Rightarrow X=\frac{2}{3}-\frac{5}{3}Y$

$\mathrm{E}(X)=\mathrm{E}\left(\frac{2}{3}-\frac{5}{3}Y\right)=\frac{2}{3}-\frac{5}{3}\mathrm{E}(Y)$

$=\frac{2}{3}-\frac{5}{3}(-0.98)=\frac{2+4.9}{3}=\frac{6.9}{3}=2.3$

$\mathrm{E}(X)=\sum x\mathrm{P}(X=x)=-2a+0\times b+2a+3b+4c=2.3$

$3b+4c=2.3$ **(2)**

$Y\geqslant-1\Rightarrow\frac{2-3X}{5}\geqslant-1$

$\Rightarrow 2-3X\geqslant-5$

$\Rightarrow X\leqslant\frac{7}{3}$

So $\mathrm{P}(Y\geqslant-1)=0.4\Rightarrow\mathrm{P}\left(X\leqslant\frac{7}{3}\right)=0.4$

$\Rightarrow 2a+b=0.4$ **(3)**

17 b Subtract equation (**3**) from equation (**1**) gives:
$b + c = 0.6$ (**4**)

Multiply equation (**4**) by 3
$3b + 3c = 1.8$ (**5**)

Subtract equation (**5**) from equation (**2**)
$c = 2.3 - 1.8 = 0.5$

Substitute value for c in equation (**4**)
$b + 0.5 = 0.6 \Rightarrow b = 0.1$

Substitute value for b in equation (**3**)
$2a + 0.1 = 0.4 \Rightarrow a = 0.15$

Solution:
$a = 0.15,\ b = 0.1,\ c = 0.5$

c $\mathrm{P}(-2X > 10Y) = \mathrm{P}(-2X > 2 \times (2 - 3X))$
$= \mathrm{P}(4X > 4) = \mathrm{P}(X > 1)$
$= a + b + c = 0.15 + 0.1 + 0.5 + 0.75$

Challenge

$$\mathrm{E}(X) = 1\times\frac{1}{n} + 2\times\frac{1}{n} + 3\times\frac{1}{n} + 4\times\frac{1}{n} + \ldots\ldots + n\times\frac{1}{n}$$

$$= \frac{1}{n}\left(1+2+3+4+\ldots\ldots+n\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} i$$

$$= \frac{1}{n}\times\frac{n(n+1)}{2} \quad \text{(using first sum in hint)}$$

$$= \frac{n+1}{2}$$

$$\mathrm{E}(X^2) = \frac{1}{n}\left(1+4+9+16+\ldots\ldots+n^2\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} i^2$$

$$= \frac{1}{n}\times\frac{n(n+1)(2n+1)}{6} \quad \text{(using second sum in hint)}$$

$$= \frac{(n+1)(2n+1)}{6}$$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{2(n+1)(2n+1)}{12} - \frac{3(n+1)^2}{12}$$

$$= \frac{4n^2+6n+2-3n^2-6n-3}{12} \quad \text{(multiplying out)}$$

$$= \frac{n^2-1}{12} \quad \text{(simplifying terms)}$$

$$= \frac{(n+1)(n-1)}{12} \quad \text{(factoring)}$$