Discrete random variables, 1C

1 a The probability distribution for Y where $Y = 2X - 3$ is:

y	–1	1	3	5
P(*Y* = *y*)	0.1	0.3	0.2	0.4

b $\mathrm{E}(Y) = \sum y\mathrm{P}(Y = y)$

$= -1 \times 0.1 + 1 \times 0.3 + 3 \times 0.2 + 5 \times 0.4$

$= -0.1 + 0.3 + 0.6 + 2$

$= 2.8$

c $\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$

$= 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.4$

$= 0.1 + 0.6 + 0.6 + 1.6$

$= 2.9$

$\mathrm{E}(2X - 3) = \mathrm{E}(Y) = 2.8$

$2\mathrm{E}(X) - 3 = 2 \times 2.9 - 3 = 5.8 - 3 = 2.8$

So $\mathrm{E}(2X - 3) = 2\mathrm{E}(X) - 3$

2 a The probability distribution for Y where $Y = X^3$ is:

y	–8	–1	0	1	8
P(*Y* = *y*)	0.1	0.1	0.2	0.4	0.2

b $\mathrm{E}(Y) = \sum y\mathrm{P}(Y = y)$

$= -8 \times 0.1 + (-1) \times 0.1 + 0 \times 0.2 + 1 \times 0.4 + 8 \times 0.2$

$= -0.8 - 0.1 + 0 + 0.4 + 1.6$

$= 1.1$

3 a $\mathrm{E}(8X) = 8\mathrm{E}(X) = 8$

b $\mathrm{E}(X + 3) = \mathrm{E}(X) + 3 = 1 + 3 = 4$

c $\mathrm{Var}(X + 3) = \mathrm{Var}(X) = 2$

d $\mathrm{Var}(3X) = 3^2\mathrm{Var}(X) = 3^2 \times 2 = 9 \times 2 = 18$

e $\mathrm{Var}(1 - 2X) = (-2)^2\mathrm{Var}(X) = 4 \times 2 = 8$

f $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

So $2 = \mathrm{E}(X^2) - 1^2$

$\Rightarrow \mathrm{E}(X^2) = 3$

4 a $\mathrm{E}(2X) = 2\mathrm{E}(X) = 2 \times 3 = 6$

b $\mathrm{E}(3 - 4X) = 3 - 4\mathrm{E}(X) = 3 - 4 \times 3 = -9$

c $\mathrm{E}(X^2 - 4X) = \mathrm{E}(X^2) - \mathrm{E}(4X)$

$= \mathrm{E}(X^2) - 4\mathrm{E}(X)$

$= 10 - 4 \times 3 = -2$

d $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 10 - 3^2 = 1$

e $\mathrm{Var}(3X + 2) = 3^2\mathrm{Var}(X) = 9\mathrm{Var}(X) = 1$

5 **a** $E(4X) = 4E(X) = 4\mu$

b $E(2X + 2) = 2E(X) + 2 = 2\mu + 2$

c $E(2X - 2) = 2E(X) - 2 = 2\mu - 2$

d The standard deviation of a random variable is the square root of its variance.

So if the standard deviation is σ, the variance is σ^2.

$\text{Var}(2X + 2) = 2^2\,\text{Var}(X) = 4\sigma^2$

e $\text{Var}(2X - 2) = 2^2\,\text{Var}(X) = 4\sigma^2$

6 **a** $E(X) = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = 3.5$

b $Y = 200 + 100X$

c $E(Y) = E(200 + 100X) = 200 + 100E(X)$

$= 200 + 100 \times 3.5 = 200 + 350 = £550$ $= 200 + 350 = £550$

7 Assume the pizzas are cylindrical and that the amount of pizza dough is given by the volume of the cylinder. The volume of a cylinder is $\pi r^2 h$. The volumes of the different sizes of pizza are then:

Size	**Small**	**Medium**	**Large**
Radius (cm)	10	15	20
Volume (cm^3)	$\pi \times 10^2 \times 1 = 100\pi$	$\pi \times 15^2 \times 1 = 225\pi$	$\pi \times 20^2 \times 1 = 400\pi$

Let the amount of pizza dough be the discrete random variable V.
The probability distribution for V is:

Size	**Small**	**Medium**	**Large**
v	100π	225π	400π
$P(V = v)$	$\frac{3}{10}$	$\frac{9}{20}$	$\frac{5}{20}$

$$E(V) = \sum vP(V = v)$$
$$= 100\pi \times \frac{3}{10} + 225\pi \times \frac{9}{20} + 400\pi \times \frac{5}{20}$$
$$= (30 + 101.25 + 100)\pi = 231.25\pi$$
$$= 726.5 \text{ cm}^3 \text{ (1 d.p.)}$$

8 a This sample space diagram shows the 16 possible outcomes:

Difference between scores	**1**	**2**	**3**	**4**
1	0	1	2	3
2	1	0	1	2
3	2	1	0	1
4	3	2	1	0

Use the table to construct the probability distribution of X:

x	0	1	2	3
$\mathrm{P}(X=x)$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{2}{16}$

$$\mathrm{E}(X)=\sum x\mathrm{P}(X=x)$$
$$=0\times\frac{4}{16}+1\times\frac{6}{16}+2\times\frac{4}{16}+3\times\frac{2}{16}=\frac{20}{16}=\frac{5}{4}=1.25$$
$$\mathrm{E}(X^2)=\sum x^2\mathrm{P}(X=x)$$
$$=0^2\times\frac{4}{16}+1^2\times\frac{6}{16}+2^2\times\frac{4}{16}+3^2\times\frac{2}{16}=\frac{40}{16}=\frac{5}{2}=2.5$$
$$\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$$
$$=2.5-(1.25)^2=2.5-1.5625=0.9375$$

8 b The probability distribution for Y where $Y = 2^X$ is:

x	0	1	2	3
y	1	2	4	8
P(*Y* = *y*)	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

$$\text{E}(Y) = \sum y\text{P}(Y = y)$$
$$= 1 \times \frac{1}{4} + 2 \times \frac{3}{8} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} = \frac{24}{8} = 3$$

The probability distribution for Z where $Z = \frac{4X + 1}{2}$ is:

x	0	1	2	3
z	0.5	2.5	4.5	6.5
P(*Z* = *z*)	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

$$\text{E}(Z) = \sum z\text{P}(Z = z)$$
$$= 0.5 \times \frac{1}{4} + 2.5 \times \frac{3}{8} + 4.5 \times \frac{1}{4} + 6.5 \times \frac{1}{8}$$
$$= 0.125 + 0.9375 + 1.125 + 0.8125 = 3$$

So E(Y) = E(Z) = 0.3

c $$\text{E}(Z^2) = \sum z^2\text{P}(Z = z)$$
$$= (0.5)^2 \times \frac{1}{4} + (2.5)^2 \times \frac{3}{8} + (4.5)^2 \times \frac{1}{4} + (6.5)^2 \times \frac{1}{8}$$
$$= 0.0625 + 2.34375 + 5.0625 + 5.28125 = 12.75$$

$$\text{Var}(Z) = \text{E}(Z^2) - (\text{E}(Z))^2 = 12.75 - 3^2 = 3.75$$

Alternatively:

$$Z = \frac{4X + 1}{2} \quad \Rightarrow \quad Z = 2X + 0.5$$
$$\text{Var}(Z) = 2^2\,\text{Var}(X) = 4\text{Var}(X) = 4 \times 0.9375 = 3.75$$

Challenge

$$\begin{aligned}\text{LHS} &= \text{E}((X-\text{E}(X))^2)\\ &= \text{E}(X^2 - 2X\text{E}(X) + (\text{E}(X))^2)\\ &= \text{E}(X^2) - \text{E}(2X\text{E}(X)) + \text{E}((\text{E}(X))^2)\end{aligned}$$

$\text{E}(k) = k$

so $\text{E}(2X\text{E}(X)) = 2\text{E}(X)\text{E}(X) = 2(\text{E}(X))^2$

and $\text{E}((\text{E}(\text{X}))^2) = (\text{E}(\text{X}))^2$

Substituting back into the equation above gives

$$\begin{aligned}\text{LHS} &= \text{E}(X^2) - \text{E}(2X\text{E}(X)) + \text{E}((\text{E}(X))^2)\\ &= \text{E}(X^2) - 2(\text{E}(X))^2 + (\text{E}(X))^2\\ &= \text{E}(X^2) - (\text{E}(X))^2\\ &= \text{RHS}\end{aligned}$$