Discrete random variables 1B

1 a By symmetry $\mathrm{E}(X)=1$

Alternatively, use $\mathrm{E}(X)=\sum x\,\mathrm{P}(X=x)$

$$\mathrm{E}(X)=\frac{1}{5}(-1+0+1+2+3)=\frac{1}{5}\times 5=1$$

b $\mathrm{E}(X^2)=\sum x^2\,\mathrm{P}(X=x)$

$$\mathrm{E}(X^2)=\frac{1}{5}(1+0+1+4+9)=\frac{1}{5}\times 15=3$$

$$\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$$
$$=3-1^2=2$$

2 a $\mathrm{E}(X)=\sum x\,\mathrm{P}(X=x)$

$$=1\times\frac{1}{3}+2\times\frac{1}{2}+3\times\frac{1}{6}$$
$$=\frac{1}{3}+1+\frac{1}{2}=\frac{11}{6}=1.833\text{ (3 d.p.)}$$

$\mathrm{E}(X^2)=\sum x^2\,\mathrm{P}(X=x)$

$$=1\times\frac{1}{3}+4\times\frac{1}{2}+9\times\frac{1}{6}$$
$$=\frac{1}{3}+2+\frac{3}{2}=\frac{23}{6}$$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$

$$=\frac{23}{6}-\left(\frac{11}{6}\right)^2=\frac{138}{36}-\frac{121}{36}=\frac{17}{36}=0.472\text{ (3 d.p.)}$$

b $\mathrm{E}(X)=\sum x\,\mathrm{P}(X=x)$

$$=-1\times\frac{1}{4}+0\times\frac{1}{2}+1\times\frac{1}{4}=0\text{ (or derive answer by symmetry)}$$

$\mathrm{E}(X^2)=\sum x^2\,\mathrm{P}(X=x)$

$$=1\times\frac{1}{4}+0\times\frac{1}{2}+1\times\frac{1}{4}=\frac{1}{2}=0.5$$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2=0.5-0^2=0.5$

c $\mathrm{E}(X)=\sum x\,\mathrm{P}(X=x)$

$$=(-2)\times\frac{1}{3}+(-1)\times\frac{1}{3}+1\times\frac{1}{6}+2\times\frac{1}{6}$$
$$=-1+\frac{1}{2}=-\frac{1}{2}=-0.5$$

$\mathrm{E}(X^2)=\sum x^2\,\mathrm{P}(X=x)$

$$=4\times\frac{1}{3}+1\times\frac{1}{3}+1\times\frac{1}{6}+4\times\frac{1}{6}$$
$$=\frac{5}{3}+\frac{5}{6}=\frac{15}{6}=2.5$$

$\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2=2.5-(0.5)^2=2.5-0.25=2.25$

3 The probability distribution for Y is:

y	1	2	3	4	5	6	7	8
P($Y = y$)	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

$$\mathrm{E}(Y) = \frac{1}{8}(1+2+3+4+5+6+7+8) = \frac{1}{8} \times 36 = 4.5$$

$$\mathrm{E}(Y^2) = \frac{1}{8}(1+4+9+16+25+36+49+64) = \frac{1}{8} \times 204 = 25.5$$

$$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - (\mathrm{E}(Y))^2 = 25.5 - (4.5)^2 = 25.5 - 20.25 = 5.25$$

4 a This sample space diagram shows the 36 possible outcomes:

+	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Use the table to construct the probability distribution of S:

s	2	3	4	5	6	7	8	9	10	11	12
P($S = s$)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

4 b $E(S) = \frac{1}{36}(2\times1+3\times2+4\times3+5\times4+6\times5+7\times6+8\times5+9\times4+10\times3+11\times2+12\times1)$

$= \frac{1}{36}(2+6+12+20+30+42+40+36+30+22+12)$

$= \frac{252}{36} = 7$

The answer can also be derived by symmetry.

4 c $\text{E}(S^2) = \frac{1}{36}(4 + 9\times 2 + 16\times 3 + 25\times 4 + 36\times 5 + 49\times 6 + 64\times 5 + 81\times 4 + 100\times 3 + 121\times 3 + 144)$

$= \frac{1}{36}(4 + 18 + 48 + 100 + 180 + 294 + 320 + 324 + 300 + 242 + 144)$

$= \frac{1974}{36} = 54.833$ (3 d.p.)

$\text{Var}(S) = \text{E}(S^2) - (\text{E}(S))^2$

$= \frac{1974}{36} - (7)^2 = \frac{1974}{36} - 49 = \frac{1974 - 1764}{36}$

$= \frac{210}{36} = 5.833$ (3 d.p.)

d Standard deviation $= \sqrt{5.8333} = 2.415$ (3 d.p.)

5 a This sample space diagram shows the 16 possible outcomes:

Difference between scores	**1**	**2**	**3**	**4**
1	0	1	2	3
2	1	0	1	2
3	2	1	0	1
4	3	2	1	0

Use the table to construct the probability distribution of D:

d	0	1	2	3
$\text{P}(D = d)$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{2}{16}$

Simplify the probabilities:

d	0	1	2	3
$\text{P}(D = d)$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

b $\text{E}(D) = 0\times\frac{1}{4} + 1\times\frac{3}{8} + 2\times\frac{1}{4} + 3\times\frac{1}{8} = \frac{10}{8} = \frac{5}{4} = 1.25$

c $\text{E}(D^2) = 0\times\frac{1}{4} + 1\times\frac{3}{8} + 4\times\frac{1}{4} + 9\times\frac{1}{8} = \frac{20}{8} = \frac{5}{2} = 2.5$

$\text{Var}(D) = \text{E}(D^2) - (\text{E}(D))^2$

$= 2.5 - (1.25)^2 = 2.5 - 1.5625 = 0.9375$

Alternatively, in fractional form

$\text{Var}(D) = \frac{5}{2} - \left(\frac{5}{4}\right)^2 = \frac{5}{2} - \frac{25}{16} = \frac{40}{16} - \frac{25}{16} = \frac{15}{16}$

6 a $\text{P(heads on first spin)} = \frac{1}{2} \Rightarrow \text{P}(T = 1) = \frac{1}{2}$

$\text{P(tails on first spin, heads on second spin)} = \frac{1}{2} \times \frac{1}{2} \Rightarrow \text{P}(T = 2) = \frac{1}{4}$

$\text{P}(T = 3) = 1 - \left(\text{P}(T = 1) + \text{P}(T = 2)\right) = 1 - \left(\frac{1}{2} + \frac{1}{4}\right) = \frac{1}{4}$

Alternatively note that

$$\text{P}(T = 3) = P(\text{heads, heads, tails}) + P(\text{heads, heads, heads})$$
$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

b $\text{E}(T) = 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{4} = \frac{7}{4} = 1.75$

$\text{E}(T^2) = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 9 \times \frac{1}{4} = \frac{15}{4} = 3.75$

$\text{Var}(T) = \frac{15}{4} - \left(\frac{7}{4}\right)^2 = \frac{60}{16} - \frac{49}{16} = \frac{11}{16} = 0.6875$

7 a $\text{E}(X) = \sum x\text{P}(X = x) = a + 2b + 3a = 4a + 2b$

b $\sum p(x) = 1, \text{ so } 2a + b = 1$ **(1)**

As $\text{E}(X) = 4a + 2b = 2(2a + b)$

$\Rightarrow \text{E}(X) = 2$

$\text{E}(X^2) = a + 4b + 9a = 10a + 4b$

$\text{Var}(X) = E(X^2) - (E(X))^2$

$= 10a + 4b - 2^2 = 10a + 4b - 4$

As Var(X) = 0.75, this gives

$10a + 4b = 4.75$ **(2)**

Multiply equation (**1**) by 4 to give

$8a + 4b = 4$ **(3)**

Subtract (**3**) from (**2**)

$2a = 4.75 - 4 = 0.75 \Rightarrow a = 0.375$

Substitute value of a in (**1**)

$0.75 + b = 1 \Rightarrow b = 0.25$