Discrete random variables 1A

1 a The probability distribution for X^2 is:

x	2	4	6	8
P($X = x$)	0.3	0.3	0.2	0.2
x^2	4	16	36	64
P($X^2 = x^2$)	0.3	0.3	0.2	0.2

Note that for this variable $\mathrm{P}(X = x) = \mathrm{P}(X^2 = x^2)$ as X only takes positive values.

$$\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$$
$$= 2\times0.3+4\times0.3+6\times0.2+8\times0.2 = 4.6$$

$$\mathrm{E}(X^2) = \sum x^2\mathrm{P}(X = x)$$
$$= 4\times0.3+16\times0.3+36\times0.2+64\times0.2 = 26$$

b The probability distribution for X is:

x	–2	–1	1	2
P($X = x$)	0.1	0.4	0.1	0.4
x^2	4	1	1	4

In this case, X can take negative values, so calculate the values of $\mathrm{P}(X^2 = x^2)$.

$$\mathrm{P}(X^2 = 1) = \mathrm{P}(X = -1) + \mathrm{P}(X = 1) = 0.4 + 0.1 = 0.5$$
$$\mathrm{P}(X^2 = 4) = \mathrm{P}(X = -2) + \mathrm{P}(X = 2) = 0.1 + 0.4 = 0.5$$

The probability distribution for X^2 is:

x^2	1	4
P($X^2 = x^2$)	0.5	0.5

$$\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$$
$$= -2\times0.1+(-1)\times0.4+1\times0.1+2\times0.8$$
$$= 0.3$$

$$\mathrm{E}(X^2) = \sum x^2\mathrm{P}(X = x)$$
$$= 4\times0.1+1\times0.4+1\times0.1+4\times0.4$$
$$= 2.5$$

Using the X^2 distribution to calculate $\mathrm{E}(X^2)$ gives the same result

$$\mathrm{E}(X^2) = \sum x^2\mathrm{P}(X^2 = x^2) = 1\times0.5+4\times0.5 = 2.5$$

2 $\mathrm{E}(X) = \sum x\,\mathrm{P}(X = x)$

$= (1\times 0.1) + (2\times 0.1) + (3\times 0.1) + (4\times 0.2) + (5\times 0.4) + (6\times 0.1)$

$= 0.1 + 0.2 + 0.3 + 0.8 + 2.0 + 0.6$

$= 4$

$\mathrm{E}(X^2) = \sum x^2\,\mathrm{P}(X = x)$

$= (1\times 0.1) + (4\times 0.1) + (9\times 0.1) + (16\times 0.2) + (25\times 0.4) + (36\times 0.1)$

$= 0.1 + 0.4 + 0.9 + 3.2 + 10 + 3.6$

$= 18.2$

3 a The probability distribution for X is:

x	2	3	6
$\mathrm{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

The probability distribution for X^2 is:

x^2	4	9	36
$\mathrm{P}(X^2 = x^2)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

b $\mathrm{E}(X) = \sum x\,\mathrm{P}(X = x)$

$= 2\times\frac{1}{2} + 3\times\frac{1}{3} + 6\times\frac{1}{6}$

$= 1 + 1 + 1$

$= 3$

$\mathrm{E}(X^2) = \sum x^2\mathrm{P}(X = x)$

$= 4\times\frac{1}{2} + 9\times\frac{1}{3} + 36\times\frac{1}{6}$

$= 11$

c $(\mathrm{E}(X))^2 = 3^2 = 9$ and $\mathrm{E}(X^2) = 11$ from part **b**

So $(\mathrm{E}(X))^2$ does not equal $\mathrm{E}(X^2)$

4 a The probability distribution for X is:

x	1	2	3	4	5
$\mathbf{P}(X=x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

b $\mathrm{E}(X)=\sum x\,\mathrm{P}(X=x)$

$$=1\times\frac{1}{2}+2\times\frac{1}{4}+3\times\frac{1}{8}+4\times\frac{1}{16}+5\times\frac{1}{16}$$

$$=\frac{1}{2}+\frac{1}{2}+\frac{6}{16}+\frac{9}{16}=\frac{31}{16}$$

$$=1.9375$$

$\mathrm{E}(X^2)=\sum x^2\mathrm{P}(X=x)$

$$=1^2\times\frac{1}{2}+2^2\times\frac{1}{4}+3^2\times\frac{1}{8}+4^2\times\frac{1}{16}+5^2\times\frac{1}{16}$$

$$=1\times\frac{1}{2}+4\times\frac{1}{4}+9\times\frac{1}{8}+16\times\frac{1}{16}+25\times\frac{1}{16}$$

$$=5\tfrac{3}{16}=\frac{83}{16}$$

$$=5.1875$$

c $(\mathrm{E}(X))^2=(1.9375)^2=3.7539$ (4 d.p.)

So $(\mathrm{E}(X))^2$ does not equal $\mathrm{E}(X^2)$

5 The probabilities add up to 1, so

$0.1+a+b+0.2+0.1=1$

$a+b=0.6$ **(1)**

$\mathrm{E}(X)=\sum x\mathrm{P}(X=x)=2.9$, so

$(1\times0.1)+(2\times a)+(3\times b)+(4\times0.2)+(5\times0.1)=2.9$

$0.1+2a+3b+0.8+0.5=2.9$

$2a+3b=1.5$ **(2)**

Multiply (**1**) by 2

$2a+2b=1.2$ **(3)**

Subtract equation (**3**) from (**1**) to give

$b=0.3$

Substitute the value of b in equation (**3**) to obtain

$2a+0.6=1.2$

$\Rightarrow a=0.3$

6 $\text{E}(X) = \sum x\text{P}(X = x) = 0.3$, so

$$0.3 = (-2\times 0.1) + (-1\times a) + (1\times b) + (2\times c)$$
$$0.5 = -a + b + 2c \qquad \textbf{(1)}$$

$\text{E}(X^2) = \sum x^2\text{P}(X = x)$, so

$$1.9 = (4\times 0.1) + (1\times a) + (1\times b) + (4\times c)$$
$$1.5 = a + b + 4c \qquad \textbf{(2)}$$

$\sum \text{P}(X = x) = 1$, so

$$1 = 0.1 + a + b + c$$
$$0.9 = a + b + c \qquad \textbf{(3)}$$

$\textbf{(2)} - \textbf{(3)} \quad \Rightarrow \quad 0.6 = 3c$

$$c = 0.2$$

$\textbf{(3)} - \textbf{(1)} \quad \Rightarrow \quad 0.4 = 2a - c$

$$2a = 0.4 + c = 0.6$$
$$a = 0.3$$

Substituting for a in **(3)** gives

$$0.9 = 0.3 + b + 0.2$$
$$0.4 = b$$

So the full solution is

$a = 0.3, \quad b = 0.4, \quad c = 0.2$

7 The probability distribution for X is:

x	–2	–1	0	5
P($X = x$)	$3a$	$2a$	a	b

$\mathrm{E}(X) = \sum x\mathrm{P}(X = x) = 1.2$, so
$1.2 = -2 \times 3a - 1 \times 2a + 0 \times a + 5 \times b$
$1.2 = -6a - 2a + 5b$
$1.2 = -8a + 5b \qquad \mathbf{(1)}$

$\sum P(X = x) = 1$, so
$1 = 3a + 2a + a + b$
$1 = 6a + b \qquad \mathbf{(2)}$

$(\mathbf{2}) \times 5 \quad \Rightarrow \quad 5 = 30a + 5b \qquad (\mathbf{3})$

$(\mathbf{3}) - (\mathbf{1}) \ \Rightarrow 3.8 = 38a \Rightarrow a = 0.1$

Substituting for a in equation $(\mathbf{2})$ gives
$b = 1 - 6a = 1 - 0.6 = 0.4$

So the full solution is
$a = 0.1, \quad b = 0.4$

8 a Suppose the probability distribution for X is:

x	1	2	3	4	5	6
$\mathrm{P}(X=x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	a	b

$\mathrm{E}(X)=\sum x\mathrm{P}(X=x)=4.1$, so

$$4.1=1\times\frac{1}{8}+2\times\frac{1}{8}+3\times\frac{1}{8}+4\times\frac{1}{8}+5\times a+6\times b$$

$$4.1=\frac{10}{8}+5a+6b$$

$$2.85=5a+6b \qquad \textbf{(1)}$$

$\sum P(X=x)=1$, so

$$1=\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+a+b$$

$$0.5=a+b \qquad \textbf{(2)}$$

$$(\mathbf{2})\times 5 \quad\Rightarrow\quad 2.5=5a+5b \qquad \textbf{(3)}$$

$$(\mathbf{1})-(\mathbf{3}) \quad\Rightarrow\quad 0.35=b$$

Substituting for b in equation (**2**) gives

$$0.5=a+0.35\Rightarrow a=0.15$$

So the full solution is

$$a=0.15=\frac{3}{20},\quad b=0.35=\frac{7}{20}$$

So the full probability distribution for X is:

x	1	2	3	4	5	6
$\mathrm{P}(X=x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{20}$	$\frac{7}{20}$

8 b Let X = number of 6s in 10 rolls of the dice, then $X \sim \mathrm{B}\left(10, \frac{7}{20}\right)$

To answer the question requires finding $P(X \geq 3)$

$P(X \geq 3) = 1 - \mathrm{P}(X = 0) - \mathrm{P}(X = 1) - \mathrm{P}(X = 2)$

Use the formula for the probability mass function of a binomial distribution

$\mathrm{P}(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$

In this case n = 10 and $p = 0.35$

$$\mathrm{P}(X \geq 3) = 1 - \left(\left(\binom{10}{0} \times 0.35^0 \times 0.65^{10} \right) + \binom{10}{1} \times 0.35^1 \times 0.65^9 + \binom{10}{2} \times 0.35^2 \times 0.65^8 \right)$$
$$= 1 - (1 \times 1 \times 0.013463 + 10 \times 0.35 \times 0.020712 + 45 \times 0.1225 \times 0.031864)$$
$$= 1 - (0.013463 + 0.072492 + 0.17565) = 1 - 0.261605$$
$$= 0.7384 \text{ (4 d.p.)}$$

9 This is the result of each throw of the dice:

Score	1	2	3	4	5	6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
Prize (£)	0	0	0	P	P	5

Let X be the amount paid out in prize money. So, using data from the table above, the probability distribution for X is:

x	0	P	5
P($X = x$)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$ is the expected amount paid out in prize money.

$\mathrm{E}(X) = 0 \times \frac{1}{2} + P \times \frac{1}{3} + 5 \times \frac{1}{6}$

Let expected profit = 0

Expected profit = cost of game − expected amount paid out in prize money

$0 = 1 - \left(\frac{1}{3}P + \frac{5}{6} \right)$

$\Rightarrow \frac{1}{3}P = \frac{1}{6} \Rightarrow P = 0.5$

So if Jorge awards a prize of £0.50 for players throwing 4 or 5 he will break even with his game and not make a loss. The maximum value of P is £0.50.

Challenge

When three dice are thrown there are $6 \times 6 \times 6 = 216$ outcomes.

There is only one way of the number 1 being the highest score on the three dice and that is 1, 1, 1.

To achieve the highest score of 2, each dice must be either 1 or 2. So there are $2 \times 2 \times 2 = 8$ ways for the highest score on three dice to be no more than 2. But one of those is 1, 1, 1, which gives a highest score of 1 so this needs to be subtracted to leave 7 possible ways for a highest score of 2.

To achieve the highest score of 3, each dice must be either 1 or 2 or 3. So there are $3 \times 3 \times 3 = 27$ ways for the highest score on three dice to be no more than 3. But one of those is 1, 1, 1, which gives a highest score of 1 and there are 7 possible ways for a highest score of 2 so these both need to be subtracted to give 19 ways of getting a highest score of 3.

Using this approach, this is the number of ways of getting each highest score:

Highest score on the three dice	Working	Number of ways
1	1, 1, 1	1
2	$2 \times 2 \times 2 - 1$	7
3	$3 \times 3 \times 3 - 7 - 1$	19
4	$4 \times 4 \times 4 - 19 - 7 - 1$	37
5	$5 \times 5 \times 5 - 37 - 19 - 7 - 1$	61
6	$6 \times 6 \times 6 - 61 - 37 - 19 - 7 - 1$	91

Converting the number of ways into probabilities gives:

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{216}$	$\frac{7}{216}$	$\frac{19}{216}$	$\frac{37}{216}$	$\frac{61}{216}$	$\frac{91}{216}$

$$\begin{aligned} E(X) &= \sum x P(X = x) \\ &= 1 \times \frac{1}{216} + 2 \times \frac{7}{216} + 3 \times \frac{19}{216} + 4 \times \frac{37}{216} + 5 \times \frac{61}{216} + 6 \times \frac{91}{216} \\ &= \frac{1071}{216} = \frac{119}{24} = 4.9583 \text{ (4 d.p.)} \end{aligned}$$