Review Exercise 2

1 a **A** has 3 columns and **B** has 2 rows.

The number of columns in **A** is not the same as the number of rows in **B**.

Therefore, the product **AB** does not exist.

b **B** has 2 columns and **A** has 2 rows.

The number of columns in **B** is the same as the number of rows in **A**.

Therefore, the product **BA** exists:

$$\mathbf{BA} = \begin{pmatrix} q & 0 \\ 3 & -1 \end{pmatrix} \times \begin{pmatrix} 3 & 2 & p \\ 0 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3q & 2q & pq \\ 9 & 4 & 3p+1 \end{pmatrix}$$

c **BA** has 3 columns and **C** has 3 rows.

The number of columns in **BA** is the same as the number of rows in **A**.

Therefore, the product **BAC** exists:

$$\mathbf{BAC} = \begin{pmatrix} 3q & 2q & pq \\ 9 & 4 & 3p+1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 12q - 6q + pq \\ 36 - 12 + 3p + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6q + pq \\ 3p + 25 \end{pmatrix}$$

d **C** has 1 column and **B** has 2 rows.

The number of columns in **C** is not the same as the number of rows in **B**.

Therefore, the product **CBA** does not exist.

2 $\mathbf{M}^2 = \begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 0\times 0+3\times(-1) & 0\times 3+3\times 2 \\ (-1)\times 0+2\times(-1) & (-1)\times 3+2\times 2 \end{pmatrix}$

$= \begin{pmatrix} 0-3 & 0+6 \\ 0-2 & -3+4 \end{pmatrix} = \begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix}$

$\mathbf{M}^2$ is quite complicated to work out and it is sensible to calculate this before working out $\mathbf{M}^2 + a\mathbf{M} + b\mathbf{I}$

Then consider $\mathbf{M}^2 + a\mathbf{M} + b\mathbf{I} = \mathbf{O}$

$\begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix} + a\begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3a \\ -a & 2a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} -3+b & 6+3a \\ -2-a & 1+2a+b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Equating the top left elements

$-3+b=0 \Rightarrow b=3$

Equating the top right elements

$6+3a=0 \Rightarrow a=-2$

$a=-2,\ b=3$

There are four elements which could be equated but you only need to equate two of them to find a and b. You could use the others to check your working. For example; if $a = -2$, $b = 3$ then $1 + 2a + b = 1 - 4 + 3$ which does equal 0.

3 $\mathbf{A}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$

$\mathbf{A}^2 - (a+d)\mathbf{A}$

$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} - \begin{pmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{pmatrix}$

$= \begin{pmatrix} a^2+bc-a^2-ad & ab+bd-ab-bd \\ ac+cd-ac-cd & bc+d^2-ad-d^2 \end{pmatrix}$

$= \begin{pmatrix} bc-ad & 0 \\ 0 & bc-ad \end{pmatrix} = \lambda\mathbf{I} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so

$\lambda\mathbf{I} = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

You can write down the results of simple calculations like this without showing all of the working.

Equating the top left (or bottom right elements)

$\lambda = bc - ad$

Note that $\lambda = -\det(\mathbf{A})$.

4 $\mathbf{A}^2 = \begin{pmatrix} 1 & 2 & b \\ 3 & 0 & 1 \\ a & -1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & b \\ 3 & 0 & 1 \\ a & -1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 1\times1+2\times3+b\times a & 1\times2+2\times0+b\times(-1) & 1\times b+2\times1+b\times2 \\ 3\times1+0\times3+1\times a & 3\times2+0\times0+1\times(-1) & 3\times b+0\times1+1\times2 \\ a\times1+(-1)\times3+2\times a & a\times2+(-1)\times0+2\times(-1) & a\times b+(-1)\times1+2\times2 \end{pmatrix}$$

$$= \begin{pmatrix} ab+7 & 2-b & 3b+2 \\ 3+a & 5 & 3b+2 \\ 3a-3 & 2a-2 & ab+3 \end{pmatrix}$$

Compare corresponding elements to given $\mathbf{A}^2$:

$$\begin{pmatrix} ab+7 & 2-b & 3b+2 \\ 3+a & 5 & 3b+2 \\ 3a-3 & 2a-2 & ab+3 \end{pmatrix} = \begin{pmatrix} 3 & 3 & -1 \\ 7 & 5 & -1 \\ 9 & 6 & -1 \end{pmatrix}$$

$2a-2=6$ $\quad$ $2-b=3$

$2a=8$ $\quad$ $\Rightarrow b=-1$

$\Rightarrow a=4$

5 a $\det(\mathbf{A}) = 2\times(-1) - 3\times p = -2-3p$

If **A** is singular, $\det(\mathbf{A}) = 0$.

$-2-3p=0 \Rightarrow 3p=-2 \Rightarrow p=-\frac{2}{3}$

> You need to know that, if $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

b As in part (a), $\det(\mathbf{A}) = -2-3p$

$-2-3p=4 \Rightarrow -3p=6 \Rightarrow p=-2$

c $\mathbf{A}^2 = \begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 4-6 & 6-3 \\ -4+2 & -6+1 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -2 & -5 \end{pmatrix}$

$\mathbf{A}^2 - \mathbf{A} = \begin{pmatrix} -2 & 3 \\ -2 & -5 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}$

$= \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = -4\mathbf{I}$

This is the required result with $k = -4$.

6 a For a matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Here, $\det(\mathbf{A}) = 4\times 2 - (-1)\times(-6) = 8 - 6 = 2$

$$\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix}$$

You need to remember this property of the inverse of matrices. The order of **A** and **B** is reversed in this formula.

b $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

$$= \begin{pmatrix} 2 & 0 \\ 3 & p \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 3p+3 & 2p+\frac{3}{2} \end{pmatrix}$$

c $(\mathbf{AB})(\mathbf{AB})^{-1} = \mathbf{I}$

The product of any matrix and its inverse is **I**. This applies to a product matrix, **AB** in this case, as well as to a matrix such as **A**.

$$\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3p+3 & 2p+\frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Equating the upper left elements

Finding all four of the elements of the product matrix of the left hand side of this equation would be lengthy. To find p, you only need one equation, so you only need to consider one element. Here the upper left hand element has been used but you could choose any of the four elements.

$$-1\times 2 + 2(3p+3) = 1$$
$$-2 + 6p + 6 = 1$$
$$6p = -3$$
$$p = -\frac{1}{2}$$

7 a

$$\det \mathbf{A} = k\begin{vmatrix} -1 & k \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & k \\ 9 & 0 \end{vmatrix} + (-2)\begin{vmatrix} 0 & -1 \\ 9 & 1 \end{vmatrix}$$

The 2 × 2 determinants are worked out using the formula $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, which you learnt in book FP1.

$$= k(-k) - 1\times(-9k) + (-2)\times 9$$
$$= -k^2 + 9k - 18 = 0$$
$$k^2 - 9k + 18 = (k-3)(k-6) = 0$$
$$k = 3, 6$$

A singular matrix is a matrix without an inverse. The determinant of a singular matrix is 0.

7 b The matrix of the minors, **M** say, is given by

$$\mathbf{M}=\begin{pmatrix} \begin{vmatrix}-1 & k\\ 1 & 0\end{vmatrix} & \begin{vmatrix}0 & k\\ 9 & 0\end{vmatrix} & \begin{vmatrix}0 & -1\\ 9 & 1\end{vmatrix} \\ \begin{vmatrix}1 & -2\\ 1 & 0\end{vmatrix} & \begin{vmatrix}k & -2\\ 9 & 0\end{vmatrix} & \begin{vmatrix}k & 1\\ 9 & 1\end{vmatrix} \\ \begin{vmatrix}1 & -2\\ -1 & k\end{vmatrix} & \begin{vmatrix}k & -2\\ 0 & k\end{vmatrix} & \begin{vmatrix}k & 1\\ 0 & -1\end{vmatrix}\end{pmatrix}$$

$$=\begin{pmatrix}-k & -9k & 9\\ 2 & 18 & k-9\\ k-2 & k^2 & -k\end{pmatrix}$$

As you have worked out the determinant of **A** in part **a**, the remaining steps for working out an inverse of a 3 × 3 matrix are:
1 Work out the matrix of the minors.
2 Obtain the matrix of cofactors by adjusting the signs of the minors using the pattern

$$\begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix}$$

3 Transpose the matrix of the cofactors.
4 Divide the transpose of the matrix of cofactors by the determinant of the matrix.

The matrix of the cofactors, **C** say, is given by

$$\mathbf{C}=\begin{pmatrix}-k & 9k & 9\\ -2 & 18 & -k+9\\ k-2 & -k^2 & -k\end{pmatrix}$$

The transpose of the matrix of the cofactors is given by

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}-k & -2 & k-2\\ 9k & 18 & -k^2\\ 9 & -k+9 & -k\end{pmatrix}$$

The inverse of **A** is given by

$$\mathbf{A}^{-1}=\frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathrm{T}}$$

$$=\frac{1}{-k^2+9k-18}\begin{pmatrix}-k & -2 & k-2\\ 9k & 18 & -k^2\\ 9 & -k+9 & -k\end{pmatrix}$$

You have worked out the determinant of **A** in part **a**. It is perfectly acceptable to leave your answer in this form. You do not have to divide every individual term in the matrix by $-k^2+9k-18$.

8 $\mathbf{A}=\begin{pmatrix}2p & p & 2\\ 3 & 0 & 0\\ -1 & 1 & -1\end{pmatrix}$

$$\det \mathbf{A}=2p\begin{vmatrix}0 & 0\\ 1 & -1\end{vmatrix}-p\begin{vmatrix}3 & 0\\ -1 & -1\end{vmatrix}+2\begin{vmatrix}3 & 0\\ -1 & 1\end{vmatrix}$$
$$=2p(0-0)-p(-3-0)+2(3-0)$$
$$=3p+6=3(p+2)$$

$$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}0 & 0\\ 1 & -1\end{vmatrix} & \begin{vmatrix}3 & 0\\ -1 & -1\end{vmatrix} & \begin{vmatrix}3 & 0\\ -1 & 1\end{vmatrix}\\ \begin{vmatrix}p & 2\\ 1 & -1\end{vmatrix} & \begin{vmatrix}2p & 2\\ -1 & -1\end{vmatrix} & \begin{vmatrix}2p & p\\ -1 & 1\end{vmatrix}\\ \begin{vmatrix}p & 2\\ 0 & 0\end{vmatrix} & \begin{vmatrix}2p & 2\\ 3 & 0\end{vmatrix} & \begin{vmatrix}2p & p\\ 3 & 0\end{vmatrix}\end{pmatrix}$$
$$=\begin{pmatrix}0 & -3 & 3\\ -p-2 & -2p+2 & 3p\\ 0 & -6 & -3p\end{pmatrix}$$

$$\mathbf{C}=\begin{pmatrix}0 & 3 & 3\\ p+2 & -2p+2 & -3p\\ 0 & 6 & -3p\end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}0 & p+2 & 0\\ 3 & -2p+2 & 6\\ 3 & -3p & -3p\end{pmatrix}$$

$$\mathbf{A}^{-1}=\frac{1}{3(p+2)}\begin{pmatrix}0 & p+2 & 0\\ 3 & -2p+2 & 6\\ 3 & -3p & -3p\end{pmatrix}$$

9 **a** $\det(\mathbf{A}) = 4p \times q - (-q) \times (-3p)$

$= 4pq - 3pq = pq$

$\mathbf{A}^{-1} = \frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}$

In your final answer, you could multiply each term in the matrix by $\frac{1}{pq}$, which would give

$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{p} & \frac{1}{p} \\ \frac{3}{q} & \frac{4}{q} \end{pmatrix}$

b $\mathbf{AX} = \begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$

Multiply both sides on the left by $\mathbf{A}^{-1}$

$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$

$\mathbf{X} = \frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$

$= \frac{1}{pq}\begin{pmatrix} 2pq - pq & 3q^2 + q^2 \\ 6p^2 - 4p^2 & 9pq + 4pq \end{pmatrix}$

$= \frac{1}{pq}\begin{pmatrix} pq & 4q^2 \\ 2p^2 & 13pq \end{pmatrix}$

It is important to multiply by $\mathbf{A}^{-1}$ on the correct side of the expression. As shown here, multiplying on the left of **AX**, you get $\mathbf{A}^{-1}\mathbf{AX} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{IX} = \mathbf{X}$, which is what you are asked to find.

If instead you multiplied both sides on the right by $\mathbf{A}^{-1}$ you would get $\mathbf{AXA}^{-1}$, which does not simplify, and no further progress can be made.

Working out $\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}\frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}$ instead of $\frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$ is a common error.

10 Write the system of equations using matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 3 & -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \qquad (*)$$

Find the inverse of the left-hand matrix, **A**:

$$\det\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = -14$$

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -1 & -2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 8 & -1 \\ 3 & -2 & -5 \\ -1 & -4 & -3 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} -5 & -8 & -1 \\ -3 & -2 & 5 \\ -1 & 4 & -3 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -5 & -3 & -1 \\ -8 & -2 & 4 \\ -1 & 5 & -3 \end{pmatrix}$$

$$\mathbf{A}^{-1} = -\frac{1}{14}\begin{pmatrix} -5 & -3 & -1 \\ -8 & -2 & 4 \\ -1 & 5 & -3 \end{pmatrix}$$

Multiplying both sides of the matrix equation (*) on the left by $\mathbf{A}^{-1}$ gives

$$\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{I}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{14}\begin{pmatrix} -5 & -3 & -1 \\ -8 & -2 & 4 \\ -1 & 5 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{14}\begin{pmatrix} -14 \\ -28 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

The single point of intersection is (1, 2, 0).

11 Let w be the initial number of woolly llamas, c the initial number of classic llamas and S the initial number of Suri llamas.

Flock initially has 2810:

$\Rightarrow w+c+s=2810$ (1)

160 more woolly llamas than classic:

$\Rightarrow w-c=160$ (2)

Given increase after one year:

$1.05w+1.03c+0.96S=2810+46=2856$ (3)

Write equations (1), (2) and (3) using matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1.05 & 1.03 & 0.96 \end{pmatrix}\begin{pmatrix} w \\ c \\ S \end{pmatrix}=\begin{pmatrix} 2810 \\ 160 \\ 2856 \end{pmatrix} \quad (*)$$

Find the inverse of the left-hand matrix, **A**:

$$\det\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1.05 & 1.03 & 0.96 \end{vmatrix}=0.16$$

$$\mathbf{M}=\begin{pmatrix} \begin{vmatrix} -1 & 0 \\ 1.03 & 0.96 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1.05 & 0.96 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1.05 & 1.03 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 1.03 & 0.96 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1.05 & 0.96 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1.05 & 1.03 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \end{pmatrix}$$

$$=\begin{pmatrix} -0.96 & 0.96 & 2.08 \\ -0.07 & -0.09 & -0.02 \\ 1 & -1 & -2 \end{pmatrix}$$

$$\mathbf{C}=\begin{pmatrix} -0.96 & -0.96 & 2.08 \\ 0.07 & -0.09 & 0.02 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix} -0.96 & 0.07 & 1 \\ -0.96 & -0.09 & 1 \\ 2.08 & 0.02 & -2 \end{pmatrix}$$

$$\mathbf{A}^{-1}=\frac{1}{0.16}\begin{pmatrix} -0.96 & 0.07 & 1 \\ -0.96 & -0.09 & 1 \\ 2.08 & 0.02 & -2 \end{pmatrix}$$

11 (cont.)

Multiplying both sides of the matrix equation (*) on the left by $\mathbf{A}^{-1}$ gives

$$\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} w \\ c \\ S \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 2810 \\ 160 \\ 2856 \end{pmatrix}$$

$$\mathbf{I}\begin{pmatrix} w \\ c \\ S \end{pmatrix} = \frac{1}{0.16}\begin{pmatrix} -0.96 & 0.07 & 1 \\ -0.96 & -0.09 & 1 \\ 2.08 & 0.02 & -2 \end{pmatrix}\begin{pmatrix} 2810 \\ 160 \\ 2856 \end{pmatrix}$$

$$\begin{pmatrix} w \\ c \\ S \end{pmatrix} = \frac{1}{0.16}\begin{pmatrix} 169.6 \\ 144 \\ 136 \end{pmatrix} = \begin{pmatrix} 1060 \\ 900 \\ 850 \end{pmatrix}$$

Initially there were 1060 woolly llamas, 900 classic llamas and 850 Suri llamas.

12 a $\begin{pmatrix} 1 & -2 & -p \\ 2 & p & 5 \\ 1 & 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ p \\ -p \end{pmatrix}$

planes do not meet, so $\begin{pmatrix} 1 & -2 & -p \\ 2 & p & 5 \\ 1 & 3 & -2 \end{pmatrix}$ has no inverse

Hence $\det\begin{vmatrix} 1 & -2 & -p \\ 2 & p & 5 \\ 1 & 3 & -2 \end{vmatrix} = 0$

$0 = 1(-2p - 15) - (-2)(-4 - 5) - p(6 - p)$

$0 = -2p - 15 - 18 - 6p + p^2$

$0 = p^2 - 8p - 33$

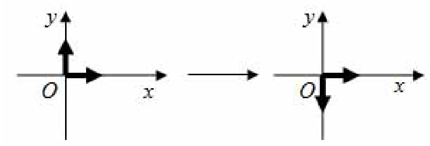
$0 = (p + 3)(p - 11)$

$p = -3$ or $p = 11$

b $p = -3$ the system is consistent and the planes form a sheaf.

$p = 11$ the system is inconsistent and the planes form a prism.

13 a Reflection in the x axis transforms



$\begin{pmatrix}1\\0\end{pmatrix}\rightarrow\begin{pmatrix}1\\0\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix}\rightarrow\begin{pmatrix}0\\-1\end{pmatrix}$

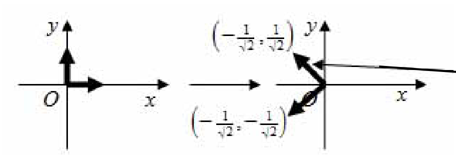
(1, 0) lies on the x-axis and so is not changed by reflection in the x-axis.

So

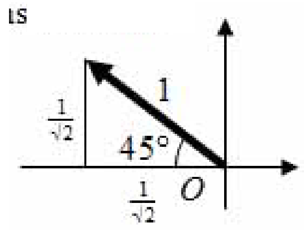
$$\mathbf{A}=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

Unless the question states otherwise, it is acceptable to write down a simple matrix like this without showing your working.

Rotation of +135° about (0,0) transforms



The geometry of the vector to which $\begin{pmatrix}1\\0\end{pmatrix}$ is transformed is



$\begin{pmatrix}1\\0\end{pmatrix}\rightarrow\begin{pmatrix}-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix}\rightarrow\begin{pmatrix}-\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{pmatrix}$

So

$$\mathbf{B}=\begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

Arrows have been added so that you can see where the columns in **B** come from.

$$\mathbf{C}=\mathbf{AB}=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

13 b $\mathbf{C}^2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$$= \begin{pmatrix} \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) \\ \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}, \text{as required.}$$

As an example of the calculations;

$$-\frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} = +\frac{1\times 1}{\sqrt{2}\times\sqrt{2}} = \frac{1}{2}$$

Alternatively, to make the calculation simpler you could initially write

$\mathbf{C} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$ and square this expression.

14 a $\mathbf{M}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Equating the elements:

$a = 3, c = 2$

Also,

$\mathbf{M}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Using $a = 3, c = 2$ from above,

$$\begin{pmatrix} 3 & b \\ 2 & d \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6+b \\ 4+d \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Equating the upper elements

$6 + b = 2 \Rightarrow b = -4$

Equating the lower elements

$4 + d = 1 \Rightarrow d = -3$

$a = 3, b = -4, c = 2, d = -3$

In questions about transformations, you need to write the coordinates of points as column vectors. For example, the coordinate (1, 0) is written as the column vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

14 b $\mathbf{M} = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}$

$\mathbf{M}^2 = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 9-8 & -12+12 \\ 6-6 & -8+9 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$, as required

c As $\mathbf{M}^2 = \mathbf{I}, \mathbf{M}^{-1}\mathbf{M}^2 = \mathbf{M}^{-1}\mathbf{I}$

$\mathbf{M} = \mathbf{M}^{-1}$

The matrix **M** is its own inverse. This follows from the result in part **b**. In more detail:

$$\mathbf{M}^2 = \mathbf{I}$$
$$\mathbf{MM} = \mathbf{I}$$
$$\mathbf{M}^{-1}(\mathbf{MM}) = \mathbf{M}^{-1}\mathbf{I}$$
$$(\mathbf{M}^{-1}\mathbf{M})\mathbf{M} = \mathbf{M}^{-1}$$
$$\mathbf{IM} = \mathbf{M}^{-1}$$
$$\mathbf{M} = \mathbf{M}^{-1}$$

$\mathbf{M}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$

$\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix} p \\ q \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} 8 \\ -3 \end{pmatrix}$

$\mathbf{I}\begin{pmatrix} p \\ q \end{pmatrix} = \mathbf{M}\begin{pmatrix} 8 \\ -3 \end{pmatrix}$

In this question, as **M** is its own inverse, you can replace $\mathbf{M}^{-1}$ by $\mathbf{M}$.

$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 8 \\ -3 \end{pmatrix} = \begin{pmatrix} 24+12 \\ 16+9 \end{pmatrix} = \begin{pmatrix} 36 \\ 25 \end{pmatrix}$

Hence $p = 36, q = 25$

15 a $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 2y-x \\ 3y \end{pmatrix} = \begin{pmatrix} -1x+2y \\ 0x+3y \end{pmatrix}$

$= \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

So $\mathbf{C} = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$

b $\begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 3x \\ 6x \end{pmatrix}$; $6x = 2(3x)$ so the point satisfies the equation of the original line.

15 c $\det(\mathbf{C}) = -1\times 3 - 2\times 0 = -3$

$$\mathbf{C}^{-1} = \frac{1}{-3}\begin{pmatrix} 3 & -2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$$

You are given the results of transforming the points by T and are asked to find the original points. You are "working backwards" to the original points and so you will need the inverse matrix.

Let the coordinates of A, B and C be

$(x_A, y_A), (x_B, y_B)$ and (x_C, y_C) respectively.

$$\mathbf{C}\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$$

$$\mathbf{C}^{-1}\mathbf{C}\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \mathbf{C}^{-1}\begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$$

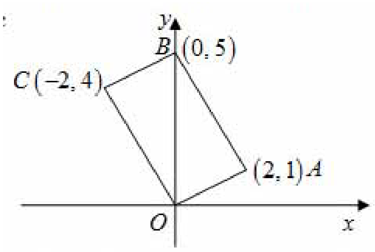
$$\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \begin{pmatrix} -1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -10+10 & -10+8 \\ 1 & 5 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & -2 \\ 1 & 5 & 4 \end{pmatrix}$$

Hence $A:(2,1),\ B:(0,5),\ C:(-2,4)$

15 d



Considering the gradients of the sides

$$m_{OA} = \frac{1}{2};\; m_{CB} = \frac{5-4}{0-(-2)} = \frac{1}{2}$$

So OA is parallel to CB.

$$m_{OC} = \frac{4-0}{-2-0} = -2;\quad m_{AB} = \frac{5-1}{0-2} = \frac{4}{-2} = -2$$

So OC is parallel to AB.
The opposite sides of $OABC$ are parallel to each other and so $OABC$ is a parallelogram.
To show that $OABC$ is specifically a rectangle (not just any parallelogram), we must show that one interior angle is a right-angle:
Consider the angle AOC

$$m_{OA} \times m_{OC} = \frac{1}{2} \times -2 = -1.$$

So OA is perpendicular to OC, and hence AOC is a right angle.
So the parallelogram $OABC$ contains a right angle, and hence $OABC$ is a rectangle.

Using the properties of quadrilaterals you learnt for GCSE, there are many alternative ways of showing that $OABC$ is a rectangle. This is just one of many possibilities, using the result you learnt in the C1 module that the gradient of the line joining (x_1, y_1) to (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

16 a

$$\det \mathbf{A} = 3\begin{vmatrix}1 & 1\\3 & u\end{vmatrix} - 1\begin{vmatrix}1 & 1\\5 & u\end{vmatrix} + (-1)\begin{vmatrix}1 & 1\\5 & 3\end{vmatrix}$$
$$= 3(u-3) - 1(u-5) - 1\times(-2)$$
$$= 3u - 9 - u + 5 + 2$$
$$= 2u - 2$$
$$= 2(u-1), \text{as required}$$

Each 2×2 determinant is evaluated using the formula $\begin{vmatrix}a & b\\c & d\end{vmatrix} = ad - bc$.

16 b The matrix of the minors, **M** say, is given by

$$\mathbf{M}=\begin{pmatrix} \begin{vmatrix}1 & 1\\ 3 & u\end{vmatrix} & \begin{vmatrix}1 & 1\\ 5 & u\end{vmatrix} & \begin{vmatrix}1 & 1\\ 5 & 3\end{vmatrix} \\ \begin{vmatrix}1 & -1\\ 3 & u\end{vmatrix} & \begin{vmatrix}3 & -1\\ 5 & u\end{vmatrix} & \begin{vmatrix}3 & 1\\ 5 & 3\end{vmatrix} \\ \begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix} & \begin{vmatrix}3 & -1\\ 1 & 1\end{vmatrix} & \begin{vmatrix}3 & 1\\ 1 & 1\end{vmatrix} \end{pmatrix}$$

$$=\begin{pmatrix} u-3 & u-5 & -2 \\ u+3 & 3u+5 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

The minor of an element is found by deleting the row and the column in which the element lies.
For example, to find the minor of b in $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, delete the row and column through b $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.
The minor is the determinant of the elements left, that is $\begin{vmatrix} d & f \\ g & i \end{vmatrix}$.

The matrix of the cofactors, **C** say, is given by

$$\mathbf{C}=\begin{pmatrix} u-3 & -u+5 & 2 \\ -u-3 & 3u+5 & -4 \\ 2 & -4 & 2 \end{pmatrix}$$

The transpose of the matrix of the cofactors is given by

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix} u-3 & -u-3 & 2 \\ -u+5 & 3u+5 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

The inverse of **A** is given by

$$\mathbf{A}^{-1}=\frac{1}{\det \mathbf{A}}\mathbf{C}^{\mathrm{T}}$$

$$=\frac{1}{2(u-1)}\begin{pmatrix} u-3 & -u-3 & 2 \\ -u+5 & 3u+5 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

16 c Substituting $u = 6$, $\mathbf{A}^{-1}$ becomes:

$$\mathbf{A}^{-1} = \frac{1}{10}\begin{pmatrix} 3 & -9 & 2 \\ -1 & 23 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

The matrix you're given in part **c** is the matrix **A**, used in parts **a** and **b**, with $u = 6$.

To find the object vector when you are given the image vector, you will need the inverse matrix, $\mathbf{A}^{-1}$, with $u = 6$.

$$\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

This equation expresses the information that the image of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, under the transformation represented by the matrix **A**, is $\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$.

Multiplying both sides on the left by $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

Hence, as $\mathrm{AA}^{-1} = \mathbf{I}$,

$$\mathbf{I}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 3 & -9 & 2 \\ -1 & 23 & -4 \\ -2 & -4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 9-9+12 \\ -3+23-24 \\ -6-4+12 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 12 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1.2 \\ -0.4 \\ 0.2 \end{pmatrix}$$

$\therefore a = 1.2,\ b = -0.4,\ c = 0.2$

17 a

$$\mathbf{MM} = \mathbf{I}$$

By definition, $\mathbf{MM}^{-1} = \mathbf{I}$.
As you have been given that $\mathbf{M} = \mathbf{M}^{-1}$, it follows that $\mathbf{MM} = \mathbf{I}$. The matrix is self-inverse.

$$\begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 9+2a & 3a+ab & 0 \\ 6+2b & 2a+b^2 & 0 \\ 4c & ac & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If two matrices are equal, then all of the corresponding elements in the matrices must be equal. Potentially, there are 9 equations here. However, since this question has only 3 unknowns (a, b and c), you need to pick out 3 equations which would be most convenient to solve for a, b and c.

Equating the first elements in the first row

$9 + 2a = 1 \Rightarrow a = -4$

Equating the first elements in the second row

$6 + 2b = 0 \Rightarrow b = -3$

Equating the first elements in the third row

$4c = 0 \Rightarrow c = 0$

17 b Using the values of a, b, and c found in part **a**

$$\mathbf{M} = \begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det \mathbf{M} = 3\begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix} - (-4)\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & -3 \\ 0 & 0 \end{vmatrix}$$

$$= 3 \times (-3) + 4 \times 2 = -1$$

c Let a point which is invariant under R have position vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The vector of an invariant point is unchanged when multiplied by the matrix representing the transformation.

$$\begin{pmatrix} 3x-4y \\ 2x-3y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating the top elements

$3x - 4y = x \Rightarrow 2x - 4y = 0 \Rightarrow x = 2y$

Equating the middle elements

$2x - 3y = y \Rightarrow 2x - 4y = 0 \Rightarrow x = 2y$

Equating the top elements and the middle elements gives the same equation, $x = 2y$.
Equating the bottom elements gives $z = z$. This is an identity, always satisfied, and gives you no further information.
So all points satisfying $x = 2y$ remain invariant under R

An equation satisfied by all the invariant points is $x = 2y$.

Because the transformation R is 3-dimensional, $x = 2y$ represents a plane of points.

18 a $\det(\mathbf{A}) = k \times 2k - (k-1) \times (-3)$

$= 2k^2 + 3k - 3$

b The triangle has been enlarged by a factor of

$\frac{198}{18} = 11$

So $\det(\mathbf{A}) = 11$

$2k^2 + 3k - 3 = 11$

$2k^2 + 3k - 14 = 0$

$(2k+7)(k-2) = 0$

$k = -\frac{7}{2}, 2$

The determinant is the area scale factor in transformations. This is equivalent to $\frac{\text{area of image}}{\text{area of object}} = \det(\mathbf{A})$. So the scale factor is part **b** must equal the determinant in part **a**.

19 a $\det\mathbf{M} = \begin{vmatrix} \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{vmatrix}$

$= \frac{3}{\sqrt{2}} \times \frac{3}{\sqrt{2}} - \left(-\frac{3}{\sqrt{2}}\right) \times \frac{3}{\sqrt{2}}$

$= \frac{9}{2} + \frac{9}{2} = 9$

Area scale factor $= 9$

Scale factor $= \sqrt{9} = 3$

For a linear transformation represented by matrix **M**, det **M** represents the scale factor for the change in area.
Hence $\sqrt{\det \mathbf{M}}$ represents the linear scale factor for the enlargement.

b $\begin{pmatrix} \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} 3\cos\theta & -3\sin\theta \\ 3\sin\theta & 3\cos\theta \end{pmatrix}$

$3\cos\theta = \frac{3}{\sqrt{2}}$

$\cos\theta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\theta = 45^\circ$ anti-clockwise about $(0,0)$

Checking using the elements gives **M**.

19 c Let the coordinates of P be (x, y)

Applying the matrix **M** to $\begin{pmatrix} x \\ y \end{pmatrix}$ gives $\begin{pmatrix} p \\ q \end{pmatrix}$

$$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Left multiply by the inverse $\mathbf{M}^{-1}$

$$\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} p \\ q \end{pmatrix}$$

$$\mathbf{I}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{9}\begin{pmatrix} \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ -\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3p}{9\sqrt{2}} + \frac{3q}{9\sqrt{2}} \\ -\frac{3p}{9\sqrt{2}} + \frac{3q}{9\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{p+q}{3\sqrt{2}} \\ \frac{-p+q}{3\sqrt{2}} \end{pmatrix}$$

The coordinates of P are $\left(\frac{p+q}{3\sqrt{2}}, \frac{-p+q}{3\sqrt{2}}\right)$

20 Take a general point on the line, $(y, k - \frac{1}{2}x)$ and transform it by T

$$\begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} x \\ k - \frac{1}{2}x \end{pmatrix} = \begin{pmatrix} 2k - x \\ \frac{1}{2}x \end{pmatrix}$$

$k - \frac{1}{2}(2k - x) = k - k + \frac{1}{2}x = \frac{1}{2}x$ k – 1 so the transformed point satisfies the equation of the original line.

21 $\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$

Let $n=1$.

The left-hand side becomes

$$\sum_{r=1}^{1} r(r+3) = 1(1+3) = 4$$

> $\sum_{r=1}^{1} r(r+3)$ consists of just one term.
>
> That is $r(r+3)$ with 1 substituted for *r*.

The right-hand side becomes

$$\frac{1}{3} \times 1(1+1)(1+5) = \frac{1}{3} \times 2 \times 6 = 4$$

The left-hand side and the right-hand side are equal and so the summation is true for $n=1$.

Assume the summation is true for $n=k$.

That is $\sum_{r=1}^{k} r(r+3) = \frac{1}{3}k(k+1)(k+5)\ldots \quad \ldots *$

> This is often called the **induction hypothesis**.

$$\sum_{r=1}^{k+1} r(r+3) = \sum_{r=1}^{k} r(r+3) + (k+1)(k+4)$$

> The sum from 1 to *k* + 1 is the sum from 1 to *k* plus one extra term.
>
> In this case, the extra term is found by replacing each *r* in $r(r+3)$ by $k+1$.

$$= \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4), \text{using} *$$

$$= \frac{1}{3}k(k+1)(k+5) + \frac{3}{3}(k+1)(k+4)$$

$$= \frac{1}{3}(k+1)[k(k+5) + 3(k+4)]$$

> Multiplying out the brackets would give you an awkward cubic expression which would be difficult to factorise. You should try to simplify the working by looking for any common factors and taking them outside a bracket. Here $k+1$ is a common factor.

$$= \frac{1}{3}(k+1)[k^2 + 5k + 3k + 12]$$

$$= \frac{1}{3}(k+1)[k^2 + 8k + 12]$$

$$= \frac{1}{3}(k+1)(k+2)(k+6)$$

$$= \frac{1}{3}(k+1)((k+1)+1)((k+1)+5)$$

> This expression is $\frac{1}{3}n(n+1)(n+5)$ with each *n* replaced by $k+1$.

This is the result obtained by substituting $n=k+1$ into the right-hand side of the summation and so the summation is true for $n=k+1$.

The summation is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the summation is true for all positive integers *n*.

22 $\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

Let $n=1$.

The left-hand side becomes

$$\sum_{r=1}^{1}(2r-1)^2 = (2-1)^2 = 1^2 = 1$$

$\sum_{r=1}^{1}(2r-1)^2$ consists of just one term. That is $(2r-1)^2$ with 1 substituted for r.

The right-hand side becomes

$$\frac{1}{3}\times 1(2-1)(2+1) = \frac{1}{3}\times 1\times 1\times 3 = 1$$

The left-hand side and the right-hand side are equal and so the summation is true for $n=1$.

Assume the summation is true for $n=k$.

That is $\sum_{r=1}^{k}(2r-1)^2 = \frac{1}{3}k(2k-1)(2k+1)\ldots\ldots *$

$$\sum_{r=1}^{k+1}(2r-1)^2 = \sum_{r=1}^{k}(2r-1)^2 + (2(k+1)-1)^2$$

The sum from 1 to $k+1$ is the sum from 1 to k plus one extra term. In this case, the extra term is found by replacing the r in $(2r-1)^2$ by $k+1$.

$$= \sum_{r=1}^{k}(2r-1)^2 + (2k+1)^2$$

$$= \frac{1}{3}k(2k-1)(2k+1) + \frac{3}{3}(2k+1)^2, \text{ using } *$$

$$= \frac{1}{3}(2k+1)[k(2k-1)+3(2k+1)]$$

Multiplying out the brackets would give you an awkward cubic expression which would be difficult to factorise. Look for any common factors and take them outside a bracket. Here $(2k+1)$ is a common factor.

$$= \frac{1}{3}(2k+1)[2k^2+5k+3]$$

$$= \frac{1}{3}(2k+1)(k+1)(2k+3)$$

$$= \frac{1}{3}(k+1)(2(k+1)-1)(2(k+1)+1)$$

This expression is $\frac{1}{3}n(2n-1)(2n+1)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n=k+1$ into the right-hand side of the summation and so the summation is true for $n=k+1$.

The summation is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the summation is true for all positive integers n.

23 $\sum_{r=1}^{n} a_r = \sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$

Let $n = 1$.

The left-hand side becomes

$$\sum_{r=1}^{1} r(r+1)(2r+1) = 1 \times 2 \times 3 = 6$$

The right-hand side becomes

$$\frac{1}{2} \times 1 \times 2^2 \times 3 = 6$$

The left-hand side and the right-hand side are equal and so the summation is true for $n = 1$.

All inductions need to be shown to be true for a small number, usually 1.

Assume the summation is true for $n = k$.

That is $\sum_{r=1}^{k} r(r+1)(2r+1) = \frac{1}{2}k(k+1)^2(k+2)$ *

$$\sum_{r=1}^{k+1} r(r+1)(2r+1) = \sum_{r=1}^{k} r(r+1)(2r+1) + (k+1)(k+2)(2k+3)$$

$$= \frac{1}{2}k(k+1)^2(k+2) + \frac{2}{2}(k+1)(k+2)(2k+3), \text{using } *$$

$$= \frac{1}{2}(k+1)(k+2)[k(k+1) + 2(2k+3)]$$

$$= \frac{1}{2}(k+1)(k+2)[k^2 + 5k + 6]$$

$$= \frac{1}{2}(k+1)(k+2)(k+2)(k+3)$$

$$= \frac{1}{2}(k+1)(k+2)^2(k+3)$$

$$= \frac{1}{2}(k+1)((k+1)+1)^2((k+1)+2)$$

Fractions need to be expressed to the same denominator before factorising. The form of the answer shows that you need to have $\frac{1}{2}$ as a common factor and it helps you to write $\frac{2}{2}$ before the second term on the right-hand side of the summation.

This expression is $\frac{1}{2}n(n+1)^2(2n+1)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n = k+1$ into the right-hand side of the summation and so the summation is true for $n = k+1$.

The summation is true for $n = 1$, and if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the summation is true for all positive integers n.

24 $\sum_{r=1}^{n} r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2)$

Let $n = 1$.

The left-hand side becomes

$$\sum_{r=1}^{1} r^2(r-1) = 1^2 \times (1-1) = 0$$

$\sum_{r=1}^{1} r^2(r-1)$ consists of just one term. That is $r^2(r-1)$ with 1 substituted for r. In this case, because of the bracket, this clearly gives 0.

The right-hand side becomes

$$\frac{1}{12} \times 1 \times (1-1) \times (1+1) \times (3+2)$$

$$= \frac{1}{12} \times 1 \times 0 \times 2 \times 5 = 0$$

The left-hand side and the right-hand side are equal and so the summation is true for $n = 1$.

Assume the summation is true for $n = k$.

That is $\sum_{r=1}^{k} r^2(r-1) = \frac{1}{12}k(k-1)(k+1)(3k+2) \ldots\ldots *$

$$\sum_{r=1}^{k+1} r^2(r-1) = \sum_{r=1}^{k} r^2(r-1) + (k+1)^2(k+1-1)$$

The common factors in these two terms are $\frac{1}{12}$, k and $(k+1)$.

$$= \frac{1}{12}k(k-1)(k+1)(3k+2) + \frac{12}{12}k(k+1)^2, \text{ using } *$$

$$= \frac{1}{12}k(k+1)[(k-1)(3k+2) + 12(k+1)]$$

$$= \frac{1}{12}k(k+1)[3k^2 - k - 2 + 12k + 12]$$

$$= \frac{1}{12}k(k+1)[3k^2 + 11k + 10]$$

$$= \frac{1}{12}(k+1)k(k+2)(3k+5)$$

$$= \frac{1}{12}(k+1)((k+1)-1)((k+1)+1)(3(k+1)+2)$$

Rearrange this expression so that it is the right-hand side of the summation with n replaced by $k+1$.

This is the result obtained by substituting $n = k+1$ into the right-hand side of the summation and so the summation is true for $n = k+1$.

The summation is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the summation is true for all positive integers n.

25 a $f(n) = 3^{4n} + 2^{4n+2}$

$f(k+1) - f(k) = 3^{4(k+1)} + 2^{4(k+1)+2} - (3^{4k} + 2^{4k+2})$

$= 3^{4k+4} - 3^{4k} + 2^{4k+6} - 2^{4k+2}$

$= 3^{4k}(3^4 - 1) + 2^{4k}(2^6 - 2^2)$

$= 3^{4k} \times 80 + 2^{4k} \times 60$

$= 3^{4k-1} \times 3 \times 80 + 2^{4k} \times 60$

$= 240 \times 3^{4k-1} + 60 \times 2^{4k}$

$= 15(16 \times 3^{4k-1} + 4 \times 2^{4k})$ *

For all $k \in \mathbb{Z}^+$, $(16 \times 3^{4k-1} + 4 \times 2^{4k})$ is an integer, and, hence, $f(k+1) - f(k)$ is divisible by 15.

At this stage $f(k+1) - f(k)$ is clearly divisible by 10 (and 20), but since 80 is not a multiple of 15, to obtain that the expression is divisible by 15 you have to remove a further factor of 3 by writing 3^{4k} as $3^{4k-1} \times 3^1$.

This shows that 15 is a factor of $f(k+1) - f(k)$ and this is the equivalent to showing that $f(k+1) - f(k)$ is exactly divisible by 15. Note that the result would not be true for negative integers *k* as, for example, 4×2^{4k} would be a fraction less than one.

b Let $n = 1$

$f(1) = 3^4 + 2^6 = 81 + 64 = 145 = 5 \times 29$

So $f(n)$ is divisible by 5 for $n = 1$.

Assume that $f(k)$ is divisible by 5.

It would follow that $f(k) = 5m$, where m is an integer.

From *

$f(k+1) = f(k) + 15(16 \times 3^{4k-1} + 4 \times 2^{4k})$

$= 5m + 15(16 \times 3^{4k-1} + 4 \times 2^{4k})$

$= 5(m + 3(16 \times 3^{4k-1} + 4 \times 2^{4k}))$

So $f(k+1)$ is divisible by 5.

An expression which is divisible by 15 is certainly divisible by 5, which is all that is required for part **b**.

If both $f(k)$ and $15(16 \times 3^{4k-1} + 4 \times 2^{4k})$ are divisible by 5, then their sum, $f(k+1)$ is divisible by 5.

$f(n)$ is divisible by 5 for $n = 1$, and, if it is divisible by 5 for $n = k$, then it divisible by 5 for $n = k + 1$.

Although $f(k+1) - f(k)$ is divisible by 15, $f(n)$ is never divisible by 15 for any *n*. After completing part **a**, you might misread the question for part **b** and attempt to prove that 15 was a factor of $f(n)$. It is always necessary to read questions carefully

By mathematical induction, $f(n)$ is divisible by 5 for all $n \in \mathbb{Z}^+$.

26 a $f(n) = 24 \times 2^{4n} + 3^{4n}$

$f(n+1) - f(n)$

$= 24 \times 2^{4(n+1)} + 3^{4(n+1)} - 24 \times 2^{4n} - 3^{4n}$

This is an acceptable answer for part **a**. However, reading ahead, the question concerns divisibility by 5. So it is sensible to further work on this expression and show that it is divisible by 5.

b $f(n+1) - f(n)$

$= 24 \times 2^{4n+4} - 24 \times 2^{4n} + 3^{4n+4} - 3^{4n}$

$= 24 \times 2^{4n}(2^4 - 1) + 3^{4n}(3^4 - 1)$

$= 24 \times 2^{4n} \times 15 + 3^{4n} \times 80$

$= 5(72 \times 2^{4n} + 16 \times 3^{4n}) \ldots *$

In the middle of a question it is easy to forget that, in all inductions, you need to show that the result is true for a small number. This is usually 1 but this question asks you to show a result is true for all non-negative integers and 0 is a non-negative integer, so you should begin with 0.

Let $n = 0$

$f(0) = 24 \times 2^0 + 3^0 = 24 + 1 = 25$

So $f(n)$ is divisible by 5 for $n = 0$.

Assume that f (k) is divisible by 5. It would follow that $f(k) = 5m$, where m is an integer.

From*, substituting $n = k$ and rearranging,

$$f(k+1) = f(k) + 5(72 \times 2^{4n} + 16 \times 3^{4n})$$
$$= 5m + 5(72 \times 2^{4n} + 16 \times 3^{4n})$$
$$= 5(m + 72 \times 2^{4n} + 16 \times 3^{4n})$$

So $f(k+1)$ is divisible by 5.

$f(n)$ is divisible by 5 for $n = 0$, and, if it is divisible by 5 for $n = k$, then it divisible by 5 for $n = k + 1$.

By mathematical induction, $f(n)$ is divisible by 5 for all non-negative integers n.

27 Let $f(n) = 7^n + 4^n + 1$

The question gives no label to the function $7^n + 4^n + 1$. Since you are going to have to refer to this function a number of times in your solution, it helps if you call it f(n).

Let $n = 1$

$f(1) = 7^1 + 4^1 + 1 = 12$

12 is divisible by 6, so $f(n)$ is divisible by 6 for $n = 1$.

Consider $f(k+1) - f(k)$

This question gives you no hint to help you. With divisibility questions, it often helps to consider $f(k+1) - f(k)$ and try and show that this divides by the appropriate number, here 6. It does not always work and there are other methods which often work just as well or better.

$$\begin{aligned} f(k+1) - f(k) &= 7^{k+1} + 4^{k+1} + 1 - (7^k + 4^k + 1) \\ &= 7^{k+1} - 7^k + 4^{k+1} - 4^k \\ &= 7^k(7-1) + 4^k(4-1) \\ &= 6 \times 7^k + 3 \times 4^k \\ &= 6 \times 7^k + 3 \times 4 \times 4^{k-1} \\ &= 6(7^k + 2 \times 4^{k-1}) \ldots \ * \end{aligned}$$

So 6 is a factor of $f(k+1) - f(k)$.

Assume that f(k) is divisible by 6.

It would follow that $f(k) = 6m$, where m is an integer.

From *

$$\begin{aligned} f(k+1) &= f(k) + 6(7^k + 2 \times 4^{k-1}) \\ &= 6m + 6(7^k + 2 \times 4^{k-1}) \\ &= 6(m + 7^k + 2 \times 4^{k-1}) \end{aligned}$$

If both f(k) and $6(7^k + 2 \times 4^{k-1})$ are divisible by 6, then their sum, $f(k+1)$ is divisible by 6. You could write this down instead of the working shown here.

So $f(k+1)$ is divisible by 6.

f(n) is divisible by 6 for $n = 1$, and, if it is divisible by 6 for $n = k$, then it divisible by 6 for $n = k + 1$.

By mathematical induction, f(n) is divisible by 6 for all positive integers n.

28 Let $f(n) = 4^n + 6n - 1$

The question gives no label to the function $4^n + 6n - 1$. Since you are going to have to refer to this function a number of times in your solution, it helps if you call it f(n).

Let $n = 1$

$f(1) = 4^1 + 6 - 1 = 9$

So f(n) is divisible by 9 for $n = 1$.

Assume that f(k) is divisible by 9,

Then, for some integer m,

$f(k) = 4^k + 6k - 1 = 9m$

With divisibility questions, it often helps to consider $f(k+1) - f(k)$ and try and show that this divides by the appropriate number, here 9. This will work in this question if you choose to do it. However the method shown here is, for this question, a neat one and you need to be aware of various alternative methods. No particular method works every time.

Rearranging

$4^k = 9m - 6k + 1 \ldots *$

$$\begin{aligned} f(k+1) &= 4^{k+1} + 6(k+1) - 1 \\ &= 4 \times 4^k + 6k + 5 \\ &= 4 \times (9m - 6k + 1) + 6k + 5 \\ &= 36m - 24k + 4 + 6k + 5 \\ &= 36m - 18k + 9 \\ &= 9(4m - 2k + 1) \end{aligned}$$

Here you substitute the expression for 4^k in * for the 4^k in your expression for f(k + 1)

This is divisible by 9.

f(n) is divisible by 9 for $n = 1$, and, if it is divisible by 9 for $n = k$, then it divisible by 9 for $n = k + 1$.

By mathematical induction, f(n) is divisible by 9 for all $n \in \mathbb{Z}^+$.

29 Let $f(n) = 3^{4n-1} + 2^{4n-1} + 5$

Let $n = 1$

$f(1) = 3^3 + 2^3 + 5 = 27 + 8 + 5 = 40 = 10 \times 4$

So f (n) is divisible by 10 for $n = 1$.

Consider $f(k+1) - f(k)$

$f(k+1) - f(k)$

$= 3^{4k+3} + 2^{4k+3} - 5 - (3^{4k-1} + 2^{4k-1} - 5)$

$= 3^{4k+3} - 3^{4k-1} + 2^{4k+3} - 2^{4k-1}$

$= 3^{4k-1}(3^4 - 1) + 2^{4k-3}(2^6 - 2^2)$

$= 3^{4k-1} \times 80 + 2^{4k-3} \times 30$

$= 10(8 \times 3^{4k-1} + 3 \times 2^{4k-3}) \ldots *$

When you replace n by $k+1$ in, for example, 3^{4n-1} you get $3^{4(k+1)-1} = 3^{4k+4-1} = 3^{4k+3}$.

If we were to simplify $2^{4k+3} - 2^{4k-1}$ as far as possible, we could write $2^{4k-1}\left(2^4 - 1\right)$ in the next line.

However, $2^4 - 1 = 15$ which is not divisible by 10, so we only simplify the expression to $2^{4k-3}(2^6 - 2^2)$ which is divisible by 10.

Assume that f (k) is divisible by 10.

It would follow that $f(k) = 10m$, where m is an integer.

From *

$f(k+1) = f(k) + 10(8 \times 3^{4k-1} + 3 \times 2^{4k-3})$

$= 10m + 10(8 \times 3^{4k-1} + 3 \times 2^{4k-3})$

$= 10(m + (8 \times 3^{4k-1} + 3 \times 2^{4k-3}))$

If both f (k) and $10(8 \times 3^{4k-1} + 3 \times 2^{4k-3})$ are divisible by 10, then their sum, f (k+1) is divisible by 10. If you preferred, you could write this down instead of the working shown here.

So $f(k+1)$ is divisible by 10.

f(n) is divisible by 10 for $n = 1$, and, if it is divisible by 10 for $n = k$, then it divisible by 10 for $n = k + 1$.

By mathematical induction, f(n) is divisible by 10 for all positive integers n.

30 $\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$

Let $n = 1$

$\mathbf{A}^1 = \begin{pmatrix} 1 & (2^1 - 1)c \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 2 \end{pmatrix}$

You need to begin by showing the result is true for $n = 1$. You substitute $n = 1$ into the printed expression for $\mathbf{A}^n$ and check that you get the matrix **A** as given in the question.

This is **A**, as defined in the question, so the result is true for $n = 1$.

Assume the result is true for $n = k$.

That is $\mathbf{A}^k = \begin{pmatrix} 1 & (2^k - 1)c \\ 0 & 2^k \end{pmatrix}$…… *

$\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A}$

$= \begin{pmatrix} 1 & (2^k - 1)c \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & c + 2(2^k - 1)c \\ 0 & 2 \times 2^k \end{pmatrix}$

$= \begin{pmatrix} 1 & c + 2^{k+1}c - 2c \\ 0 & 2^{k+1} \end{pmatrix}$

$= \begin{pmatrix} 1 & (2^{k+1} - 1)c \\ 0 & 2^{k+1} \end{pmatrix}$

Keep in mind as you multiply out the matrices that you are aiming at the expression $\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$ with each n replaced by $k + 1$.

$2 \times 2^k = 2^1 \times 2^k = 2^{k+1}$ by one of the laws of indices.
You use this twice.

This is the result obtained by substituting $n = k + 1$ into the result $\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$ and so the result is true for $n = k + 1$.

The result is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k + 1$.

By mathematical induction the result is true for all positive integers n.

31 $\mathbf{A}^n = \begin{pmatrix} 2n+1 & n \\ -4n & -2n+1 \end{pmatrix}$

Let $n = 1$

$\mathbf{A}^1 = \begin{pmatrix} 2+1 & 1 \\ -4 & -2+1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

You need to begin by showing the result is true for n = 1. You substitute n = 1 into the given expression for $\mathbf{A}^n$ and check that you get the matrix **A**, as given in the question.

This is **A**, as defined in the question, so the result is true for n = 1.

Assume the result is true for n = k.

That is $\mathbf{A}^k = \begin{pmatrix} 2k+1 & k \\ -4k & -2k+1 \end{pmatrix}$

The **induction hypothesis** is the result you are asked to prove with n replaced by k.

$\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A}$

$= \begin{pmatrix} 2k+1 & k \\ -4k & -2k+1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

$= \begin{pmatrix} 3(2k+1)-4k & 2k+1-k \\ -12k-4(-2k+1) & -4k-(-2k+1) \end{pmatrix}$

$= \begin{pmatrix} 2k+3 & k+1 \\ -4k-4 & -2k-1 \end{pmatrix}$

$= \begin{pmatrix} 2(k+1)+1 & k+1 \\ -4(k+1) & -2(k+1)+1 \end{pmatrix}$

$\mathbf{A}^{k+1}$ is the matrix **A**, multiplied by itself k times, multiplied by **A** one more time. $\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A}^1 = \mathbf{A}^k.\mathbf{A}$. This is one of the index laws applied to matrices

This is the result obtained by substituting $n = k+1$ into the result $\mathbf{A}^n = \begin{pmatrix} 2n+1 & n \\ -4n & -2n+1 \end{pmatrix}$ and so the result is true for $n = k+1$.

The result is true for n = 1, and, if it is true for n = k, then it is true for n = k + 1.

By mathematical induction the result is true for all positive integers n.

32 a He has not shown the general statement to be true for $k = 1$.

32 b Let $f(n) = 2^{2n} - 1$, where $n \in \mathbb{Z}^+$

$f(1) = 2^{2(1)} - 1 = 3$, which is divisible by 3.

f(n) is divisible by 3 when $n = 1$.

Assume true for $n = k$, so that $f(k) = 2^{2k} - 1$ is divisible by 3.

$$\begin{aligned} f(k+1) &= 2^{2(k+1)} - 1 \\ &= 2^{2k} \times 2^2 - 1 \\ &= 4(2^{2k}) - 1 \\ &= 4(f(k) + 1) - 1 \\ &= 4f(k) + 3 \end{aligned}$$

Therefore f(n) is divisible by 3 when $n = k + 1$

If $f(n)$ is divisible by 3 when $n = k$, then it has been shown that $f(n)$ is also divisible by 3 when $n = k + 1$.

As $f(n)$ is divisible by 3 when $n = 1$, $f(n)$ is also divisible by 3 for all $n \in \mathbb{Z}^+$ by mathematical induction.

33 A general point on the line has position vector $\begin{pmatrix} 2-\lambda \\ -1-2\lambda \\ 3+3\lambda \end{pmatrix}$.

Let $A = \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix}$ and $B = \begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix}$

For A, when $\lambda = 4$:

$x_A = 2 - 4 \Rightarrow x_A = -2$

$y_A = -1 - 2(4) \Rightarrow y_A = -9$

$z_A = 3 + 3(4) \Rightarrow z_A = 15$

For B, when $\lambda = -1$:

$x_B = 2 + 1 \Rightarrow x_B = 3$

$y_B = -1 - 2(-1) \Rightarrow y_B = 1$

$z_B = 3 + 3(-1) \Rightarrow z_B = 0$

The distance AB is given by

$$\begin{aligned} |\overrightarrow{AB}| &= \sqrt{\left((3-(-2))^2 + (1-(-9))^2 + (0-15)^2\right)} \\ &= \sqrt{350} = 5\sqrt{14} \end{aligned}$$

34 Line through P, Q and R has direction vector $\overrightarrow{PQ} = \begin{pmatrix} a-1 \\ 4 \\ 5 \end{pmatrix}$

You could write any alternative form of the direction vector, e.g. $\overrightarrow{PR}$ or $\overrightarrow{QR}$

So the equation of the line through all 3 points can be written

as $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} a-1 \\ 4 \\ 5 \end{pmatrix}$

You could write any alternative form of the line, e.g.
$\mathbf{r} = \overrightarrow{R} + \lambda\overrightarrow{PR}$ or $\mathbf{r} = \overrightarrow{Q} + \lambda\overrightarrow{QR}$

Now R is a point on the line, so for some λ it is true that:

$$\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} a-1 \\ 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ b \end{pmatrix}$$

Equate your expression for the line with the point you haven't yet used.

Equate y-components to find λ: $-1 + 4\lambda = 7$

$4\lambda = 8 \Rightarrow \lambda = 2$

Use the x-component of $\mathbf{r}$ to find a:

$1 + 2(a-1) = 5$

$2a - 1 = 5$

$2a = 6 \Rightarrow a = 3$

Use the z-component of $\mathbf{r}$ to find b:

$3 + 2 \times 5 = b$

$b = 13$

The vector equation of the line is given by

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

35 a $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$

Let $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be perpendicular to the plane

Then $\mathbf{n}$ is perpendicular to both $\begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}$

So $\begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} . \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = 0$ and $\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} . \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} = 0$

Therefore $-2a+2b-c=0$ and $a-b-4c=0$

Choosing $c=0$ gives $a=b$

Therefore, choosing $a=1$ gives a normal vector of $\begin{pmatrix}1\\1\\0\end{pmatrix}$

The equation of the plane is $\mathbf{r.n}=\mathbf{a.n}$
Since $B(1,1,2)$ lies on the plane, you have

$$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}.\begin{pmatrix}1\\1\\0\end{pmatrix}$$

or $x+y=2$

b A normal vector is given by

$$\begin{pmatrix}1\\1\\0\end{pmatrix}$$

c $\frac{(2)}{2}+\frac{k}{2}=1$

$$1+\frac{k}{2}=1$$

$$2+k=2 \Rightarrow k=0$$

36 a Assuming that the lines do intersect:

$$\begin{pmatrix} 11+4\lambda \\ 5+2\lambda \\ 6+4\lambda \end{pmatrix} = \begin{pmatrix} 24+7\mu \\ 4+\mu \\ 13+5\mu \end{pmatrix} \quad *$$

You can write the equations of the lines in column vector form and equate them.

Rearranging gives:

$4\lambda - 7\mu = 13$ **(1)**

$2\lambda - \mu = -1$ **(2)**

$4\lambda - 5\mu = 7$ **(3)**

Equate the x, y and z components.

Solve these simultaneous equations.
(1) – **(3)** gives

$-2\mu = 6$

$\therefore \mu = -3$

Solve equations **(1)** and **(3)** simultaneously.

Substitute into **(1)** to give

$4\lambda + 21 = 13 \Rightarrow \lambda = -2$

Check that $\mu = -3$ and $\lambda = -2$ satisfy equation (2):

$2(-2) - (-3) = -4 + 3 = -1$

So l_1 and l_2 intersect.

b Substituting $\mu = -3$ and $\lambda = -2$ into * gives the coordinates of the point of intersection

$$\begin{pmatrix} 11-8 \\ 5-4 \\ 6-8 \end{pmatrix} = \begin{pmatrix} 24-21 \\ 4-3 \\ 13-15 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

Substituting λ or μ will give the point of intersection.

$\therefore (3, 1, -2)$ is point of intersection.

c The directions of the lines are

$4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ and $7\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

$$\cos\theta = \frac{(4\mathbf{i}+2\mathbf{j}+4\mathbf{k})\cdot(7\mathbf{i}+\mathbf{j}+5\mathbf{k})}{\sqrt{4^2+2^2+4^2}\sqrt{7^2+1^2+5^2}}$$

$$= \frac{28+2+20}{\sqrt{36}\sqrt{75}}$$

$$= \frac{50}{6\times 5\sqrt{3}}$$

$$= \frac{5}{3\sqrt{3}}$$

$$= \frac{5}{9}\sqrt{3}$$

Use $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$, where $\mathbf{a}$ and $\mathbf{b}$ are the direction vectors of the lines.

Simplify the surds.

37 a The line can be written as $\mathbf{r} = \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

You can write the line equation in this form.

Since A lies on the line, there is a value of λ such that $\begin{pmatrix} 4 \\ 8 \\ a \end{pmatrix} = \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

use $4 = 8 + \lambda$ *or* $8 = 12 + \lambda$

$\therefore \lambda = -4$

Equate the x or y coordinates to find λ.

Equate z-coordinates with $\lambda = -4$ to give $a = 14 - (-4) = 18$

Similarly, since B lies on the line, there is a value of λ such that $\begin{pmatrix} b \\ 13 \\ 13 \end{pmatrix} = \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$

Use $13 = 12 + \lambda$ or $13 = 14 - \lambda$

$\therefore \lambda = 1$

Use the y or z coordinates to find λ.

Equate x-coordinates with $\lambda = 1$ to give $b = 8 + 1 = 9$

b Direction $\overrightarrow{OP}$ is $\begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix}$ for some λ

and direction of l_1 is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

This is obtained from the equation of l_1.

These are perpendicular

$\therefore \begin{pmatrix} 8+\lambda \\ 12+\lambda \\ 14-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$

Use the condition for perpendicular lines with direction vectors **c** and **d**, $\mathbf{c} \cdot \mathbf{d} = 0$.

$\therefore 8 + \lambda + 12 + \lambda - (14 - \lambda) = 0$

$\therefore 3\lambda + 6 = 0$

$\Rightarrow \lambda = -2$

$\therefore$ Point P is at $(6, 10, 16)$

Substitute the value of λ into the line equation to give the coordinates of P.

c Distance $OP = \sqrt{6^2 + 10^2 + 16^2}$

$= \sqrt{392}$

$= 14\sqrt{2}$

Use the formula for magnitude of a vector.

Simplify the surd using $\sqrt{392} = \sqrt{196}\sqrt{2}$.

38 a $a(4\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}-5\mathbf{j}+3\mathbf{k}) = a(4\times1+1\times(-5)+2\times3)$
$= a(4-5+6) = 5a$

For A to lie on the plane with equation $\mathbf{r.n} = 5a$, when $\mathbf{r}$ is replaced by the position vector of A, $\mathbf{r.n}$ must give $5a$.

Hence A lies in the plane Π, as required.

b $\overrightarrow{BA} = a(4\mathbf{i}+\mathbf{j}+2\mathbf{k}) - a(2\mathbf{i}+11\mathbf{j}-4\mathbf{k})$
$= a(2\mathbf{i}-10\mathbf{j}+6\mathbf{k})$

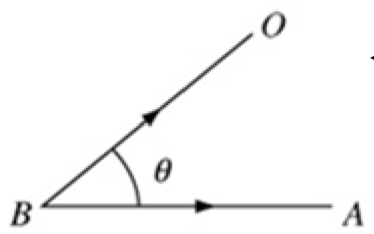
$\overrightarrow{BA} = 2a(\mathbf{i}-5\mathbf{j}+3\mathbf{k})$

$\overrightarrow{BA}$ is parallel to the vector $\mathbf{i}-5\mathbf{j}+3\mathbf{k}$, which is perpendicular to the plane Π.

This is because the equation of the plane is $\mathbf{r}.(\mathbf{i}-5\mathbf{j}+3\mathbf{k}) = 5a$ where, by definition, the vector $(\mathbf{i}-5\mathbf{j}+3\mathbf{k})$ is perpendicular to the plane.

Hence $\overrightarrow{BA}$ is perpendicular to the plane Π, as required.

c



The angle OBA is the angle between BO and BA. Both these line segments have a definite direction and so you must use the scalar product $\overrightarrow{BO}.\overrightarrow{BA}$ to find θ. If you used $\overrightarrow{OB}.\overrightarrow{BA}$, you would obtain the supplementary angle $(180° - \theta)$, which is not the correct answer

Let $\angle OAB = \theta$

$|\overrightarrow{BO}| = a\sqrt{((-2^2)+(-11)^2+4^2)} = a\sqrt{(141)}$

$|\overrightarrow{BA}| = a\sqrt{(2^2+(-10)^2+6^2)} = a\sqrt{(140)}$

$\overrightarrow{BO}.\overrightarrow{BA} = a(-2\mathbf{i}-11\mathbf{j}+4\mathbf{k}).a(2\mathbf{i}-10\mathbf{j}+6\mathbf{k})$

$$\cos\theta = \frac{\overrightarrow{BO}.\overrightarrow{BA}}{|\overrightarrow{BO}||\overrightarrow{BA}|}$$
$$= \frac{a^2((-2)\times2+(-11)\times(-10)+4\times6)}{a\sqrt{(141)}\times a\sqrt{(140)}}$$
$$= \frac{a^2(-4+110+24)}{a^2\sqrt{(141)}\sqrt{(140)}}$$
$$= \frac{130}{\sqrt{(141)}\sqrt{(140)}}$$
$$= 0.925272\ldots$$
$$\theta = 22.3°$$

39 a $\mathbf{r}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=k$ where $k=\mathbf{a}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}$ for any point in the plane with position vector **a**

Since $\begin{pmatrix}3\\4\\-1\end{pmatrix}$ is in the plane, $k=\begin{pmatrix}3\\-4\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=-10$

b Letting $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, we can write the plane as $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=-10$

So a cartesian equation of the plane is

$-x+2y+z=-10$

c Equation of l is $\mathbf{r}=\begin{pmatrix}4\\-3\\2\end{pmatrix}+t\begin{pmatrix}-1\\2\\1\end{pmatrix}$

At intersection

$$\begin{pmatrix}4-t\\-3+2t\\2+t\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=-10$$

$$-4+t-6+4t+2+t=-10$$

$$6t+2=0$$

$$6t=-2\Rightarrow t=-\frac{1}{3}$$

Then coordiantes of N are

$$4-\left(-\frac{1}{3}\right)=\frac{13}{3}$$

$$-3+2\left(-\frac{1}{3}\right)=-\frac{11}{3}$$

$$2+\left(-\frac{1}{3}\right)=\frac{5}{3}$$

$$\therefore\ N\left(\frac{13}{3},\ -\frac{11}{3},\ \frac{5}{3}\right)$$

40 a For the line l, first express each of x, y and z in terms of a third variable λ :

$$\frac{x-1}{2}=\lambda \Rightarrow x=2\lambda+1$$

$$\frac{y+3}{1}=\lambda \Rightarrow y=\lambda-3$$

$$\frac{2-z}{3}=\lambda \Rightarrow z=2-3\lambda$$

Now substitute each one into the cartesian equation of the plane:

$$2(2\lambda+1)+1(\lambda-3)-1(2-3\lambda)=5$$

$$8\lambda-3=5$$

$$8\lambda=8$$

$$\lambda=1$$

$$x=2(1)+1=3$$

$$y=(1)-3=-2$$

$$z=2-3(1)=-1$$

Position vector of P $\begin{pmatrix}3\\-2\\-1\end{pmatrix}$

40 b First, write both the plane and the line in vector form:

Plane: $2x+y-z=5$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 5$$

$$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 5 \text{ where } \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \text{ is normal to the plane}$$

Line: From part **a**, $\frac{x-1}{2} = \lambda \Rightarrow x = 2\lambda + 1$

$$\frac{y+3}{1} = \lambda \Rightarrow y = \lambda - 3$$

$$\frac{2-z}{3} = \lambda \Rightarrow z = 2 - 3\lambda$$

Hence $\mathbf{r} = (2\lambda+1)\mathbf{i} + (\lambda-3)\mathbf{j} + (2-3\lambda)\mathbf{k}$

$$\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

Let θ be the angle between the line and the *normal to the plane*:

$$\cos\theta = \frac{(2\mathbf{i}+\mathbf{j}-3\mathbf{k})\cdot(2\mathbf{i}+\mathbf{j}-\mathbf{k})}{\sqrt{2^2+1^2+(-3)^2}\sqrt{2^2+1^2+(-1)^2}}$$

$$= \frac{4+1+3}{\sqrt{14}\sqrt{6}}$$

$$= \frac{8}{\sqrt{84}}$$

$$\theta = 0.509^c$$

Then the angle between the line and the plane is: $\frac{\pi}{2} - \theta = 1.061^c$

41 a $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}, \mathbf{b} = 4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

$\therefore \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ — Use the triangle law.

$= 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$

$\therefore |\overrightarrow{AB}| = \sqrt{3^2+4^2+(-5)^2}$ — Use the formula for the magnitude of a vector.

$= \sqrt{50}$ or $5\sqrt{2}$ or 7.07

b $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

or — There are other forms of this equation, but these two are the simplest.

$\mathbf{r} = 4\mathbf{i} - 3\mathbf{j} - 2\mathbf{k} + \mu(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

41 c If $\mathbf{r} = 6\mathbf{i} + 4\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} + \mathbf{j} - \mathbf{k})$

passes through $4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

then $6 + 2\mu = 4$

$4 + \mu = 3$

$-3 - \mu = -2$

Equate *x*, *y* and *z* components.

As $\mu = -1$ satisfies all three equations, the line passes through *B* as required.

Solve for μ and check that μ satisfies all three equations.

d The lines have directions

$3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} - \mathbf{k}$

If the angle between the lines is θ then

$$\cos\theta = \frac{(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - \mathbf{k})}{|3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}||2\mathbf{i} + \mathbf{j} - \mathbf{k}|}$$

$$= \frac{3 \times 2 + 4 \times 1 + (-5) \times (-1)}{\sqrt{50}\sqrt{2^2 + 1^2 + (-1)^2}}$$

$$= \frac{15}{\sqrt{50}\sqrt{6}}$$

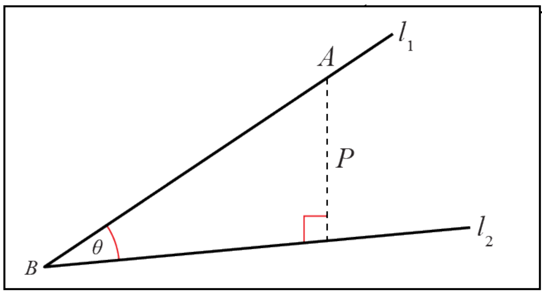
Use $\cos\theta = \frac{\mathbf{c} \cdot \mathbf{d}}{|\mathbf{c}||\mathbf{d}|}$ where **c** and **d** are the directions of the lines.

$$\therefore \cos\theta = \frac{\sqrt{3}}{2}$$

This answer is acute. If your answer is obtuse, subtract it from 180°.

and $\theta = 30°$ or $\frac{\pi}{6}$

e



Draw a diagram showing l_1, l_2 with common point *B*.

The shortest distance is the perpendicular distance.

The shortest distance from point *A* to the line l_2 is

$$|\overrightarrow{AB}|\sin\theta = 5\sqrt{2} \times \frac{1}{2}$$

$$= \frac{5\sqrt{2}}{2}$$

Use trigonometry $\sin\theta = \frac{P}{|\overrightarrow{AB}|}$.

42 a Substitute the coordinates of point P into the equation of the plane:

$-3(-1)+2(2)+3(-3)+7=5$

Distance between P and Π:

$$\frac{5}{\sqrt{(-3)^2+2^2+3^2}}=\frac{5}{\sqrt{22}}=1.066$$

b A perpendicular vector to Π is

$\mathbf{n}=-3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

Let Q have coordinates (x_1, y_1, z_1).

Let M be the midpoint of PQ.

A vector equation of the line l through P, M and Q is $\mathbf{r}=\begin{pmatrix}-1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}-3\\2\\3\end{pmatrix}$

M lies on this line so has position vector $\mathbf{r}_M=\begin{pmatrix}-1-3\lambda\\2+2\lambda\\-3+3\lambda\end{pmatrix}$ for some λ

M also lies on Π, so $\begin{pmatrix}-1-3\lambda\\2+2\lambda\\-3+3\lambda\end{pmatrix}\cdot\begin{pmatrix}-3\\2\\3\end{pmatrix}=-7$

$$3+9\lambda+4+4\lambda-9+9\lambda=-7$$
$$22\lambda-2=-7$$
$$22\lambda=-5$$
$$\lambda=-\frac{5}{22}$$

$\therefore$ M has position vector $\mathbf{r}_M=\begin{pmatrix}-1\\2\\-3\end{pmatrix}-\frac{5}{22}\begin{pmatrix}-3\\2\\3\end{pmatrix}$

P is the initial point in the equation of l, so because M has position vector $\mathbf{r}_M=\begin{pmatrix}-1\\2\\-3\end{pmatrix}-\frac{5}{22}\begin{pmatrix}-3\\2\\3\end{pmatrix}$

then Q has position vector

$$\mathbf{r}_Q=\begin{pmatrix}-1\\2\\-3\end{pmatrix}+2\times\left(-\frac{5}{22}\right)\begin{pmatrix}-3\\2\\3\end{pmatrix}=\begin{pmatrix}-1+\frac{15}{11}\\2-\frac{10}{11}\\-3-\frac{15}{11}\end{pmatrix}$$

Point Q has coordinates $\left(\frac{4}{11}, \frac{12}{11}, -\frac{48}{11}\right)$

43 a For the shark

$$\mathbf{r}_s = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2-2 \\ 11-3 \\ 11-(-1) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 8 \\ 12 \end{pmatrix}$$

For the flounder

$$\mathbf{r}_f = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$$

$2-4\lambda = 2-2\mu$ **(1)**

$3+8\lambda = -\mu$ **(2)**

$-1+12\lambda = 1+3\mu$ **(3)**

Substitute **(2)** into **(1)**:

$2-4\lambda = 2-2(-3-8\lambda)$

$2-4\lambda = 2+6+16\lambda$

$20\lambda = -6$

$\lambda = -\frac{3}{10}$

Substitute $\lambda = -\frac{3}{10}$ into **(2)**

$3+8\left(-\frac{3}{10}\right) = -\mu$

$\mu = \frac{3}{5}$

Check in **(3)**

$\text{LHS} = -1+12\left(-\frac{3}{10}\right) = -\frac{23}{5}$

$\text{RHS} = 1+3\left(\frac{3}{5}\right) = \frac{14}{5}$

$\text{LHS} \neq \text{RHS}$

Hence, the shark will never catch the flounder as their paths do not intersect.

b Unlikely that the shark will not adjust course to intercept flounder.

Challenge

1 A reflection in the plane $x = 0$ is represented by the matrix $\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Rotation, angle θ, about the y-axis is represented by the matrix $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

Now $\sin 270 = -1$ and $\cos 270 = 0$

Hence, rotation angle 270° about the y-axis is represented by the matrix $\mathbf{B} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

A reflection in the plane $y = 0$ is represented by the matrix $\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So the combination of these three transformations is given by the matrix product $\mathbf{CBA}$:

$$\mathbf{CBA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Challenge

2 Let centre P have position vector: $\mathbf{r}_p = \mathbf{c} + \lambda(\mathbf{c} - \mathbf{b}) + \mu(\mathbf{c} - \mathbf{a})$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

$\left|\overrightarrow{AP}\right|^2 = \left|\overrightarrow{CP}\right|^2$ since $\overrightarrow{AP}$ and $\overrightarrow{CP}$ are both radii of the circle which passes through A, B and C

$$(3+(2\lambda+3\mu))^2 + (4+(2\lambda+4\mu))^2 + (1+(-2\lambda+\mu))^2$$
$$= (2\lambda+3\mu)^2 + (2\lambda+4\mu)^2 + (-2\lambda+\mu)^2$$
$$26 + 6(2\lambda+3\mu) + 8(2\lambda+4\mu) + 2(-2\lambda+\mu) = 0$$
$$26 + 24\lambda + 52\mu = 0 \qquad \textbf{(1)}$$

$\left|\overrightarrow{BP}\right|^2 = \left|\overrightarrow{CP}\right|^2$ since $\overrightarrow{BP}$ and $\overrightarrow{CP}$ are both radii of the circle which passes through A, B and C

$$(2+(2\lambda+3\mu))^2 + (2+(2\lambda+4\mu))^2 + (-2+(-2\lambda+\mu))^2$$
$$= (2\lambda+3\mu)^2 + (2\lambda+4\mu)^2 + (-2\lambda+\mu)^2$$
$$12 + 4(2\lambda+3\mu) + 4(2\lambda+4\mu) - 4(-2\lambda+\mu) = 0$$
$$12 + 24\lambda + 24\mu = 0 \qquad \textbf{(2)}$$

(1) − (2): $14 + 28\mu = 0$

$$28\mu = -14 \Rightarrow \mu = -\frac{14}{28} = -\frac{1}{2}$$

Substitute $\mu = -\frac{1}{2}$ into **(2)**:

$$12 + 24\lambda + 24\left(-\frac{1}{2}\right) = 0$$
$$24\lambda = 0 \Rightarrow \lambda = 0$$

So the position vector of centre P is

$$\mathbf{r}_p = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix}$$

The coordinates of the centre of the circle are $\left(-\frac{1}{2},\ -1,\ \frac{1}{2}\right)$.

The point P is equidistant to each point A, B and C (since A, B and C lie on circumference of circle centre P). Considering point C, the radius is given by

$$\left|\overrightarrow{PC}\right| = \sqrt{\left(1-\left(-\frac{1}{2}\right)\right)^2 + (1-(-1))^2 + \left(1-\frac{1}{2}\right)^2}$$
$$= \sqrt{\frac{26}{4}}$$
$$= \sqrt{\frac{13}{2}}$$

Challenge

3 We will prove the result by induction.

$n=1$: $2(1) \le r \le \frac{1}{2}\left(1^2+1+2\right)$

$2 \le r \le 2 \Rightarrow r = 2$

It is certainly true that a single line divides a plane into two regions.
Hence the statement is true for $n=1$.

Assume that the statement is true for $n=k$:
i.e. if k non-parallel lines divide the plane into r regions, then
$2k \le r \le \frac{1}{2}\left(k^2+k+2\right)$

Now consider $k+1$ non-parallel lines

For the lower bound: when $n=k$, the plane is divided into $2k$ regions when all k lines intersect at a single point

When $n=k+1$ and all $k+1$ lines intersect at a single point there are two more regions created.

$\therefore\ r_{k+1} \geqslant 2k+2 = 2(k+1)$

For the upper bound: when $n=k$, the plane is divided into $\frac{1}{2}\left(k^2+k+2\right)$ regions when only two lines intersect at any point

When $n=k+1$ and only two lines intersect at any point, $k+1$ more regions are created

$$\begin{aligned}\therefore r_{k+1} &\leqslant \frac{1}{2}\left(k^2+k+2\right)+\left(k+1\right)\\ &= \frac{1}{2}\left(k^2+3k+4\right)\\ &= \frac{1}{2}\left(\left(k+1\right)^2+\left(k+1\right)+2\right)\end{aligned}$$

Hence $2(k+1) \leqslant r_{k+1} \leqslant \frac{1}{2}\left(\left(k+1\right)^2+\left(k+1\right)+2\right)$
If the statement holds for $n=k$, it holds for $n=k+1$.
$\therefore$ The statement holds for all positive integers, n.